Sustainability, Optimality, and Viability in the Ramsey model *

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Abstract

The Ramsey model of economic growth is revisited from the point of view of viability compared to optimality. A viable state is a state from which there exists at least one trajectory in capital, consumption, and reproduction that remains in the set of constraints of minimal consumption and positive wealth. There exists a largest set of viable states, including all others, called the *viability kernel*. This concept is an interesting addition to those of equilibria and optimal paths. Viability is first presented with a constraint of minimal consumption, then with an additional criterion of economic sustainability in the sense of the Brundtland commission, which amounts to requiring a non-decreasing social welfare. The comparison of viability kernels with or without sustainability shows how much consumption should be reduced and when. One strong mathematical result is that the viable-optimal solution in the sense of inter-temporal consumption is obtained on the viability boundary of an auxiliary system. Varying preference, technological, and demographic parameters randomly over simulated viability kernels with and without the Brundtland criterion help identify the determinants of the non-emptiness of the viability kernel and of its volume: technological progress works against population growth to favor the possibility for a given state of being viable or viable-sustainable.

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1 Introduction

The theory of economic growth is based preeminently on the neoclassical growth model, which relies on the maximization of an inter-temporal or inter-generational welfare function. The basic optimization apparatus used is optimal control as pioneered by Pontryagin. Inter-temporal optimization in traditional growth theory relies on a time discounted social welfare function, which, because it gives more weight to present generations, has no reason to satisfy long-term sustainability criteria (for the dependence of development on natural resources: Withagen, 1998; Van Geldrop and Withagen, 2000).

Beltrati et al. (1994), for example, used a standard growth model with renewable resources to show the possible non existence of any stationary long-term equilibrium for small enough discounting rates and upper-bounded renewal rate of resources. It is therefore necessary to amend the typical neoclassical framework to enforce sustainability criteria. Chichilnisky (1993) added a long-term outcome term to the traditional inter-temporal optimization of utility. This modification enables the identification of a green golden rule in the extreme case where utility is reduced to this long-term outcome term, but, in the general case of a convex combination of this term with inter-temporal utility, there is no guarantee of the existence of an optimal sustainable path (Beltratti et al., 1994). Other authors have moved away from the constant social discount rate assumption inherent in the neoclassical model and instead assumed that the discount rate is endogenous (dependent on consumption or the capital stock, as reviewed by Le Kama and Schubert (2007).

In both frameworks, inter-temporal optimization remains the criterion of inter-generational resource allocation. Arrow et al. (2004) formalized sustainability in line with the Brundtland Commission (1987), by requiring that inter-temporal social welfare $V(t)$ would not decrease over time t. Arrow et al. emphasized that this criterion "does not identify a unique consumption path: the criterion could in principle be met by many consumption paths" (: 150), and that "in defining sustainable development, there is no presumption that the consumption path being followed is in the sense of maximizing V" (: 150). The scheme advocated by these authors need not be compatible with inter-temporal optimality in the sense of optimal control, say Pontryagin optimality.

So far no criterion has been made operational. We wish to show the close affinity between sustainability as defined by the Brundtland Commission and viability theory, a theory pioneered by Nagumo in 1942 and developed by Aubin (1997). One result in viability theory is the existence and the computation of the largest (possibly empty) set, called the viability kernel, containing all initial states from which there exists at least one trajectory along which some qualitative or quantitative property —represented by a set of constraints– is satisfied up to a given, possibly infinite, time horizon. We shall draw on this theory to study Brundtland sustainability in the sense of Arrow et al. and to disentangle its economic and demographic determinants. Viability theory and the method we shall present to compute viable-optimal paths are promising when used in models incorporating energy and natural resources in the framework of environmental policy and intergenerational justice (Asheim et al., forthcoming).

In addition to containing the states through which a given criterion is optimized *while satisfying the constraints*, the viability kernel also allows one to identify sub-optimal trajectories under a given dynamic. Early applications of viability in economics are Bonneuil (1994a, 1994b), Aubin (1997), Bonneuil and Boarini (2004), Valence (2005), Marco and Romaniello (2006), and Bonneuil and Saint-Pierre (2008). The relationship between viability and optimality in the sense of optimal control, say Pontryagin optimality, has been studied in abstract settings. It receives a precise formulation in the proposal of Arrow et al. (2004). Viability theory bridges the difference between sustainability and Pontryagin optimality. Cannarsa and Frankowska (1991) showed that, for the Mayer problem, the epigraph of the value function is the viability kernel of an extended control system. We shall use similar arguments to clarify the relationship between Brundtland sustainability and Pontryagin optimality in standard growth models.

For the sake of clarity, we focus on the well-known Ramsey growth model originated in the seminal work of Frank Ramsey (1928) (Barro and Sala-i-Martin, 1995, Chapter 2). The Ramsey problem is recalled in the beginning of section 2. In this section, we also convert the Ramsey problem into a benchmark viability problem where an elementary minimal consumption constraint is to be satisfied over time. Brundtland sustainability is introduced formally. Section 2 ends with a comprehensive analysis of the viability and sustainability of optimal paths in the Ramsey model. In Section 3, elements of viability theory are presented, and the link between Pontryagin optimality and viability is made using the Ramsey model and its viability counterpart. Section 4 is devoted to the computation of viability kernels corresponding to sustainability criteria. In particular, we use the algorithm developed by Bonneuil (2006) to compute viability kernels with and without the Brundtland criterion by randomly drawing the technological, demographic and preference parameters of the model in plausible ranges. A regression of the volumes of these sets will show the influence of each determinant and overcome the intractability of obtaining an analytical expression for these volumes. We will show that sustainability is achieved at the expense of consumption, and we will identify all states from which there exists a sustainable path contained in the viability kernel. For states for which such a path no longer exists under regular dynamics, we will recommend a drastic extraordinary reduction of consumption, an impulse, outside the regular dynamics, setting the system to a viable state. Then, from this new starting point, viable policies are to be applied, changing as the state of the system navigates in the viability kernel.

2 A Preliminary investigation into Viability, Optimality, and Brundtland Sustainability

2.1 A Viability Formulation of the Ramsey Model with Minimal Consumption

The Ramsey model features a planner (either an individual or a government) whose objective is to maximize the present value of future utility gains $w(c(t))e^{-\rho t}$ as a positive function of consumption per head $c(t)$ at time t and depending on a subjective rate ρ of time preference, where $0 < \rho < 1$, over an infinite time horizon and continuous time:

$$\begin{cases} \max_{c(.)} V(0) := \max_{c(.)} \int_0^\infty w(c(\tau))e^{-\rho\tau}\, d\tau \\ \text{subject to} \\ k'(t) = f(k(t)) - (n+\delta)k(t) - c(t) \end{cases} \tag{1}$$

where "now" is time 0, n denotes the population growth rate, δ the depreciation rate of capital, $k(t)$ the capital per worker, $k(0) > 0$ is given.[1] Utility w is a strictly increasing and concave utility function, and the production function f is strictly increasing and concave. In the traditional formulation, the state constraints are:

$$k \geq 0,\, c \geq 0\,. \tag{2}$$

[1]No Ponzi game conditions are added to preclude trivial solutions of the "chain letter" type. We shall ignore this technicality and focus on more conceptual aspects of the Ramsey problem.

The optimal paths corresponding to model (1) are well known. As already mentioned, Chichilnisky (1993) criticized the discounting objective function, arguing that in many circumstances this function fails to guarantee subsistence levels for future generations. We begin with the viability idea of minimal consumption for all generations. The former objective of inter-temporal utility optimization is replaced by feasibility from the present until a given time horizon, a concept which is represented formally by the dynamics under viability constraints.

- When the capital per head $k(t)$ is governed by an autonomous differential equation, the dynamics is 2-dimensional:

$$\begin{cases} (i) & k'(t) = f(k(t)) - (n+\delta)k(t) - c(t) \\ (ii) & c'(t) = u(t) \in U := [u^\flat, u^\sharp] \end{cases} \tag{3}$$

  where U is a closed set of measurable admissible consumption changes $u(t)$, and $u^\flat \leq u^\sharp$ are two real numbers.

- When the capital per head $k(t)$ is governed by a non autonomous differential equation, notably with a production function that depends an increase in productivity over time, the dynamics is 3-dimensional:

$$\begin{cases} (i) & k'(t) = f(t,k(t)) - (n+\delta)k(t) - c(t) \\ (ii) & c'(t) = u(t) \in U := [u^\flat, u^\sharp] \\ (iii) & t' = 1 \end{cases} \tag{4}$$

with the production function defined as:

$$f(t,k(t)) := A\exp(\eta t)\hat{f}(k(t)) \tag{5}$$

with $\hat{f}$ increasing and concave, satisfying Inada conditions, A a constant expressing the technological level at time 0 and η the rate of technological progress. For the sake of simplicity, we will consider

$$\hat{f}(k) = k^{\alpha} \tag{6}$$

where α represents capital share.

In both cases, the constraints are:

$$\begin{cases} k(t) & \geq & 0 \\ c(t) & \geq & \underline{c}(t) \\ t & \in & [0, T] \end{cases} \tag{7}$$

where T is the time horizon, possibly infinite. Eq. (7) defines a set K of constraints, where $\underline{c}(t)$ is a given consumption threshold. It ensures that people consume enough to have a satisfactory standard of living. Instead of inter-temporal optimization, the programs {(3),(7)} or {(4),(7)} now require that their solutions satisfy the qualitative property (7) at any time.

One could argue that the issue of ensuring minimal consumption can be handled in an optimal control formulation (adding the corresponding control constraint). This would miss the point, made by Arrow et al. (2004) among others, that the mathematics of maintenance or sustainability differ substantially from those of optimality. We capture the issue of maintenance through the minimal consumption requirement. Adding viability constraints to the standard Ramsey model can be handled by the systematic computational analysis provided by viability theory. Optimal control experts know how difficult it is to deal with even simple state constraints.

The viability problem $\{(4),(7)\}$ has three state variables $k(t)$, $c(t)$, and t in the non autonomous case, a fact which seems to contradict the optimal control problem (1). The original control variable $c(t)$ is handled as a state variable, but no $c(0)$ value is imposed. While an optimal control technique seeks the optimal value of $c(0)$, viability is concerned with all initial conditions $(k(0), c(0))$ from which there exists at least one solution to dynamics (3) or (4) remaining in constraints (7) until a given time horizon. Another novelty with respect to optimal control is the appearance of the control $u(t)$, formalizing any admissible change in consumption, which ultimately allows us to write a state equation for $c(t)$. At any given point in time, there may be an infinity of admissible changes, namely changes in consumption compatible with viability constraints (7). At a given date, there is a technological and social constraint on consumption which, in a regular regime, rules out infinite changes in this variable. The control then varies in a bounded set U, which is more realistic than assuming that $c'(t)$ can take any value.

2.2 Brundtland Sustainability

Arrow et al. (2004) identify social welfare with "the present discounted value of the flow of utility from consumption from the present to infinity, discounted using the constant rate" $\rho > 0$ (: 149). They use the sustainability criterion from the report by the World Commission on Environment and Development (1987), known as the Brundtland Commission (after its chairperson). Sustainable development was defined as "development that meets the needs of the present without compromising the ability of future generations to meet their own needs." They take "sustainability to mean that inter-temporal social welfare $y(\theta)$ must not decrease over time θ," where θ represents the "now" which was set to 0 so far. We treat "now" as a variable in order "to concentrate on the change in V." Social welfare at time θ is then:

$$V(\theta) = \int_{\theta}^{\infty} w(c(\tau))e^{-\rho(\tau-\theta)}\,d\tau \tag{8}$$

the differential of which is:

$$V'(\theta) = -w(c(\theta)) + \rho V(\theta) \tag{9}$$

and the Brundtland condition of sustainability is:

$$V'(\theta) \geq 0 \tag{10}$$

2.3 Pontryagin Optimality, Viability, and Brundtland Sustainability: Preliminary Results

We consider the standard Ramsey model with strictly concave utility and production functions, both satisfying the Inada conditions. The Pontryagin method gives the velocities $k'(t)$ and $c'(t)$ on the optimal paths in the autonomous problem (1) (corresponding to $\eta = 0$ and $A = 1$). In this problem, the unique saddle point, where $k'(t) = 0$, $c'(t) = 0$, is $\left(\hat{k}^\star, \hat{c}^\star\right)$ with $\hat{k}^\star$ solution of $\hat{f}'_k(k) = \rho + \delta$ and $\hat{c}^\star := \hat{f}(k^\star)$, independent of the initial condition $k(0) > 0$. We define $k^*(t) = (A\exp(\eta t))^{1/(1-\alpha)}\hat{k}^*$ and $c^*(t) = (A\exp(\eta t))^{1/(1-\alpha)}\hat{c}^*$. In the autonomous problem, for any initial condition $k(0) > 0$, the level of consumption is chosen such that the system jumps on the saddle path and moves to the balanced growth point at $\left(\hat{k}^\star, \hat{c}^\star\right)$. Similarly, in the non autonomous problem (4), for any initial condition $k(0) > 0$, the level of consumption is chosen such that the system jumps on the saddle path augmented by $A\exp(\eta t))^{1/(1-\alpha)}$ and travels to the balanced growth point at $(k^\star(t), c^\star(t))$. Thus, optimal trajectories remain on a stable branch and converge to the steady state equilibrium (Barro and Sala-i-Martin, 1995, chapter 2). In the autonomous case, the equation of the stable branch in the plane (k, c) is denoted $c = \phi(k)$. In the case of a Cobb-Douglas production function and a utility function with constant elasticity of inter-temporal substitution, the most studied case in textbooks, $\phi(0) = 0$ and $\phi'(x) > 0$, $\forall x \geq 0$.[2] In particular, if $k(0) < k^\star$, consumption increases strictly from $\phi(k(0))$ to $c^\star$. If $k(0) > k^\star$, consumption is strictly decreasing from $\phi(k(0))$ to $c^\star$.

If consumption can no longer jump on the stable path, but is governed by a bounded change, and must remain above a given threshold $\underline{c}$ over time, all

[2]The concavity of the stable branch (the saddle path) depends on the inter-temporal elasticity of substitution (Barro and Sala-i-Martin, 1995, chapter 2).

states do not have the same status: from the viable ones, there exists at least one solution remaining in the set of constraints until the time horizon T; from the non viable ones, all solutions leave the set of constraints before the time horizon T.

Straightforward propositions help situate viability with respect to Pontryagin solutions. From Eq. (9),

$$V'(\theta) = \rho \int_{\theta}^{\infty} \big(- w(c(\theta)) + w(c(t))\big) e^{-\rho(t-\theta)}\, dt \tag{11}$$

On the stable branch converging to the saddle point, consumption varies monotonically, hence:

Proposition 2.1 *An optimal path converging to the saddle point and on which consumption $c(.)$ is increasing is Brundtland-sustainable in the sense that it satisfies condition (10). Optimal paths with decreasing consumption cannot be Brundtland-sustainable.*

(proof obvious).

The minimal consumption threshold, $\underline{c}(t) > 0$, is such that:

Proposition 2.2 *If, for any time t, $\underline{c}(t) > c^*(t)$, then all optimal consumption paths converging to the saddle point and starting from (k, c), $c \geq \underline{c}(0)$, leave the set of constraints in finite time, or equivalently, no such state (k, c) is viable.*

If $k(0) \leq k^*(0)$, then the optimal consumption path increases along the stable branch to $c^*(0) < \underline{c}(0)$, failing to guarantee the minimal consumption criterion during the transition. If $k(0) > k^\star(0)$, the optimal consumption path decreases to $c^\star(t)$, again failing to satisfy the minimal consumption criterion on the part of the saddle-path going from $\underline{c}(t)$ to $c^\star(t)$. We focus hereafter on the case $\underline{c}(t) < c^\star(t)$.

Proposition 2.3 *Define* $\underline{k}(t) = \phi^{-1}(\underline{c}(t))$ *such that* $0 < \underline{k}(t) < k^\star(t)$. *If* $\underline{c}(t) < c^\star(t)$, *then:*

1. *if* $0 \leq k(0) < \underline{k}(t)$, *the optimal consumption path leaves the constraints:* $(k(0), c(0))$ *is not viable.*[3]

2. *if* $\underline{k}(t) \leq k(0) \leq k^\star(t)$, *the optimal consumption path remains in the constraints and is Brundtland-sustainable:* $(k(0), c(0))$ *is viable.*

3. *if* $k(0) > k^\star(t)$, *the optimal consumption path remains in the constraints* $c(t) \geq \underline{c}(t)$ *for all* $t \geq 0$: $(k(0), c(0))$ *is viable, but not Brundtland-sustainable.*

These properties, stemming from the structure of the optimal paths in the Ramsey model and from Proposition 2.1, reveal a feature of inter-temporal optimization models: initially optimal economies starting from insufficient capital (case 1 of Proposition 2.3) cannot be viable (in the sense of guaranteeing a minimal standard of living). Optimal economies with a large enough allowance in endowment of capital (case 3) are viable, but not Brundtland-sustainable: inter-temporal substitution and income effects inherent in the standard neoclassical optimal growth

[3] In such a case, the optimal consumption paths are increasing, and Brundtland-sustainable by Proposition 1. However, this is of little interest if consumption is lower bounded.

model yield strictly decreasing optimal consumption patterns, and a subsequent decreasing net present value. The virtuous configuration where Pontryagin optimality occurs together with viability and Brundtland-sustainability lies in the intermediate case 2, where the agent consumes moderately, thus accumulating capital and continuing to increase the income of future generations.

We confirm the point made by Arrow et al. (2004), that inter-temporal optimality need not be compatible with any sustainability criterion. In this section, we showed this property for the minimal consumption criterion and for Brundtland sustainability. But if optimality in the sense of Pontryagin does not guarantee sustainability, is it possible to characterize sustainable paths in a more operational way than the mere fulfillment of state constraints (2) or (10)? This is the question we address now, in examining the relationship between optimality and viability.

3 Pontryagin Optimality and Viability: Theory

3.1 A Short Introduction to Viability Theory

Consider the autonomous problem $\{(3),(7)\}$, for n constant. It has a synthetic expression as a differential inclusion:

$$x'(t) \in F(x(t)) \text{ and } \forall t, x(t) \in K \tag{12}$$

with $K = I\!R^{+*} \times I\!R^{+}$, $x = (k, c)$ and

$$F(x) = \{(f(k) - c - (n + \delta), u) \mid u \in U\}\,. \tag{13}$$

A state $x_0 = (k_0, c_0)$ is said to be viable in K under F if there exists at least one solution $x(t)$ of Eq. (12), starting from $x(0) = x_0$ and remaining in K forever. A set of viable states is called a viability domain, and Aubin (1997) showed that there exists a maximal viability domain including all other viability domains. This set is the viability kernel $\mathrm{Viab}_F(K)$ (which is then a set of initial conditions):

$$\mathrm{Viab}_F(K) := \{x_0 \mid \exists x(.), x(0) = x_0 \text{ and } \forall t \geq 0, x'(t) \in F(x(t)), x(t) \in K\} \tag{14}$$

Trajectories visiting states outside the viability kernel are doomed to fall below their sufficiency thresholds. For viable states, some trajectories may also pass under the threshold, but there is at least one that does not. The viability property is translated into local terms through the "tangential" condition:

Theorem 3.1 (Bebernes and Schuur, 1970) *For $F : Dom(F) \subset X \mapsto X$ an upper semi-continuous correspondence with convex compact values and such that:*

$$\sup_{y \in F(x)} \parallel y \parallel \leq b(\parallel x \parallel + 1) \tag{15}$$

for some real b. K is a viability domain for F if and only if

$$\forall x \in K,\ F(x) \cap T_K(x) \neq \emptyset \tag{16}$$

where $T_K(x) = \{v \in X \text{ such that } \liminf_{h \to 0} \frac{1}{h} d_K(x + hv) = 0\}$ is the contingent cone to K at x and $d_K(x)$ the distance from x to K.

Viability conditions are stipulated at each time, not necessarily in the neighborhood of equilibria or attractors. Contrary to optimization, we are no longer concerned with predicting the trajectory the system will take, but with the maintenance of the system in K. What matters is that a right decision must be selected at the right time so as to remain in K. No knowledge of the future is required, as it is for an optimal decision. This makes viability theory valuable for dealing with sustainability.

Saint-Pierre (1994) devised an algorithm to compute this viability kernel when F is Marchaud[4] and Lipschitz. He discretized Eq. (12) so that the sequence of subsets K_j starting at $K_0 = K$ and defined recursively by:

$$K_{j+1} := K_j \cap F(K_j) \tag{17}$$

converges to a subset contained in the viability kernel of K under F. He showed that this sequence converges to the viability kernel if F is also Lipschitz:

$$\mathrm{Viab}_F(K) = \bigcap_{j=0}^{\infty} K_j \tag{18}$$

Although this algorithm is theoretically valid in any dimension, in practice, as K is reduced to a discrete grid, the algorithm must be able to update every cell of the grid at any time, which is a formidable task. The algorithm is then limited to three state dimensions. Bonneuil (2006) addressed the computation of

[4]A set-valued map $F : X \to K$ is a Marchaud map if the graph and the domain of F are closed and not empty; the values $F(x)$ are convex; F is non "explosive": $\exists b \ \forall x \in \mathrm{Dom}(F), \|F(x)\| := \max_{y \in F(x)} \|y\| \leq b(\|x\| + 1)$.

viable states and of the viability kernel in large state dimension, using a different procedure, based on stochastic optimization. The idea is to minimize the distance to the set of constraints of solutions starting from a given state, and to assess the viability status of this state whether or not the minimization of the distance leads to at least one trajectory remaining in the set of constraints. The search for viable states is also achieved by the minimization of a distance to the set of constraints, so that the procedure relies on a double stochastic optimization: one where the initial state under examination is fixed, so as to decide whether it is viable or not, and one where this initial state is varied. We shall use this algorithm later in our computational work later on.

3.2 The viability-capture basin: the non autonomous case

Consider the non autonomous problem {(4),(7)}, for n constant. It has a synthetic expression as a differential inclusion:

$$x'(t) \in F(x(t)) \text{ and } \forall t, x(t) \in K \tag{19}$$

with $x(t) = (k(t), c(t), t)$, $K = I\!R^{+3}$, and

$$F(x) = \{(f(k,t) - c - \delta k - nk, u, 1) \mid u \in U\}\,. \tag{20}$$

A state $x_0 = (k_0, c_0, 0)$ is said to be viable in $K \subset I\!R^{+2} \times \{0\}$ under F if there exists at least one solution $x(t)$ under Eq. (19), starting from $x(0) = x_0$ and that remains in K until horizon T and hits the target $C := I\!R^{+2} \times \{T\}$. The *capture-viability basin* $\mathrm{Capt}_F(K, C, T)$ at time horizon T of a target-set C viable in K

under the dynamic F is defined as the set of all states of $I\!R^{+2} \times \{0\}$ from which there exists at least one solution that remains in K until time T and hitting the target C at time T:

$$\mathrm{Capt}_F(K, C, T) := \{x_0 \mid \exists x(.), x(0) = x_0 \text{ and } \forall t \geq 0, x'(t) \in F(x(t)), x(t) \in K, x(T) \in C\} \tag{21}$$

Both Saint-Pierre's (1994) and Bonneuil's (2006) algorithms are adapted to compute capture-viability basins after the modification of the image $F(x)$ of x into:

$$\begin{cases} F(x) & \text{if } x(t) \notin C \\ \overline{\mathrm{Co}}\{F(x) \cup \{0\}\} & \text{if } x(t) \in C \end{cases} \tag{22}$$

where $\overline{\mathrm{Co}}A$ designates the closure of the smallest convex set containing the set A.

3.3 Link between Optimality and Viability

We show that optimal solutions remaining in the constraints start from the boundary of a specific capture viability basin. The classical treatment of optimality in the Ramsey model is to maximize the Hamiltonian and to focus on the behavior around the steady state. However, what about non stationary solutions which do nonetheless remain in the set of constraints?

To the classical program of inter-temporal utility maximization, we add constraints on consumption and on consumption change. We define the auxiliary system:

$$\max_{u(.)} \sup_{t \geq 0} \int_0^t w(c(\tau))e^{-\rho\tau}\, d\tau$$

subject to

$$\left\{ \begin{array}{rrcl} (i) & k'(t) & = & f(t,k(t)) - (n+\delta)k(t) - c(t) \\ (ii) & c'(t) & = & u(t) \\ (iii) & t' & = & 1 \\ (iv) & y'(t) & = & -w(c(t))e^{-\rho t} \end{array} \right. \tag{23}$$

under state constraints defining the closed set K:

$$\left\{ \begin{array}{rcl} k(t) & \geq & 0 \\ c(t) & \geq & \underline{c} \end{array} \right. \tag{24}$$

and control constraints:

$$u(t) \in U := [u^\flat, u^\sharp] \tag{25}$$

Defining $x := (k, c, V, t)$, the state of the system is $\tilde{x} := (x, y) := (k, c, V, t, y) \in \mathbb{R}^{+5}$.

Cannarsa and Frankowska (1991) showed that the infimum of $\int_0^\infty g(x(t))\, dt$ for a given positive continuous function g and a continuous-time dynamic $x'(t) \in F(x(t))$ (in continuous time) under constraints $x \in K$ for a closed set K is obtained on the boundary in the direction of low y of the viable capture basin of the set K of constraints associated with the auxiliary system:

$$\left\{ \begin{array}{rcl} x'(t) & \in & F(x(t)) \\ y'(t) & = & -g(x(t)) \end{array} \right. \tag{26}$$

where y is an auxiliary variable.

Similarly, the maximum is obtained on the boundary in the direction of high y of the viable capture basin of K associated with this auxiliary system. Some authors mistakenly believe that programming the maximum is achieved by changing a plus into a minus and the viability kernel into another specific set (the "invariance kernel"). The maximum requires a specific treatment and the interesting set is a viability-capture basin:

Proposition 3.2 *For $g : X \to I\!R^+$ a positive continuous function, the valuation function of the T-horizon control problem:*

$$\overset{T^{\sup}}{V}(x) = \sup_{u(.) \text{ solution to } (26)} \int_0^T g(x(\tau))) \, d\tau \tag{27}$$

with $x(0) = x$, is related to the capture viability basin $\mathrm{Capt}_{(26)}(K \times I\!R^+, K, T)$ *by:*

$$\overset{T^{\sup}}{V}(x) = \sup_{(x,y) \in \mathrm{Capt}_{(26)}(K \times I\!R^+, K, T)} y \tag{28}$$

Proof: For a given time horizon T, $\forall (x_1, y_1) \in \mathrm{Capt}_{(26)}(K \times I\!R^+, K, T)$, $\exists u(.)$ and $x(.)$ such that $x(0) = x_1$, $\forall t \geq 0$, $x(t) \in K$, and $y(t) = y_1 - \int_0^t g(x(\tau)) \, d\tau \geq 0$. At time T, the target $C = K$ is hit, so that

$$y(T) = 0 \tag{29}$$

and

$$y_1 = \int_0^T g(x(\tau)) \, d\tau \tag{30}$$

Hence,

$$y_1 \leq \overset{T^{\sup}}{V}(x) \tag{31}$$

or

$$\sup_{(x,y)\in \mathrm{Capt}_{(26)}(K\times I\!R^{+},K,T)} y \leq \overset{T^{\sup}}{V}(x) \tag{32}$$

Conversely, take a sequence $(\epsilon_n)_{n\in \boldsymbol{N}}$ of strictly positive real numbers converging to zero. Then, by definition of the supremum, we can associate a sequence of solution $(x_n(.))_{n\in \boldsymbol{N}}$:

$$\begin{array}{l} \forall \epsilon_n > 0, \exists x_n(.) \ \mid\ x_n(0) = x \text{ and } \forall t, x_n(t) \in K \\ \text{and } \overset{T^{\sup}}{V}(x) - \epsilon_n \leq \int_0^T g(x_n(\tau))\, d\tau \leq \overset{T^{\sup}}{V}(x) \end{array} \tag{33}$$

If $\overset{T^{\sup}}{V}(x)$ is finite, $\int_0^T g(x_n(\tau))\ d\tau$ converges. As g is continuous and K closed, $x_n(.)$ converges in K, to a solution $x(.)$. By letting ϵ_n tend to zero,

$$\overset{T^{\sup}}{V}(x) \leq \int_0^T g(x(\tau))\, d\tau \tag{34}$$

hence

$$\overset{T^{\sup}}{V}(x) = \int_0^T g(x(\tau))\, d_t au \tag{35}$$

with

$$\forall t,\ x(t) \in K \text{ and } x(0) = x \tag{36}$$

Subsequently,

$$(x, \overset{T^{\sup}}{V}(x)) \in \mathrm{Capt}_{(26)}(K \times I\!R^{+}, K, T) \tag{37}$$

□

The proposition identifies optimal paths that remain in the set of constraints. An analytic expression for the sets involved is unlikely to be available, because of non linearity and set-valued analysis, and except in trivial cases the solution is computational.

Proposition 3.2 extends to the infinite horizon optimal control problem.

Proposition 3.3 *For $g : X \to I\!R^+$ a positive continuous function, the valuation function in the infinitesimal horizon control problem:*

$$\overset{\infty\text{sup}}{V}(x) = \sup_{u(.)\text{ solution to }(26)} \int_0^\infty g(x(\tau)))\, d\tau \tag{38}$$

with $x(0) = x$, is related to the capture viability basin

$$\mathrm{Capt}_{(26)}(K \times I\!R^+, K) := \bigcup_{T \geq 0} \mathrm{Capt}_{(26)}(K \times I\!R^+, K, T) \tag{39}$$

by:

$$\overset{\infty\text{sup}}{V}(x) = \sup_{(x,y)\in \mathrm{Capt}_{(26)}(K\times I\!R^+,K)} y \tag{40}$$

proof: As g is positive, $\forall x \in X, \forall T \geq 0, \overset{\infty\text{sup}}{V}(x) \geq \overset{T\text{sup}}{V}(x)$. From Proposition 3.2, $\forall T \geq 0$,

$$\forall T \geq 0, \overset{\infty\text{sup}}{V}(x) \geq \sup_{(x,y)\in \mathrm{Capt}_{(26)}(K\times I\!R^+,K,T)} y \tag{41}$$

or

$$\overset{\infty\text{sup}}{V}(x) \geq \sup_{(x,y)\in \bigcup_{T\geq 0} \mathrm{Capt}_{(26)}(K\times I\!R^+,K,T)} y \tag{42}$$

or

$$\overset{\infty\text{sup}}{V}(x) \geq \sup_{(x,y)\in \mathrm{Capt}_{(26)}(K\times I\!R^+,K)} y \tag{43}$$

Conversely, take a sequence $(\epsilon_n)_{n\in I\!N}$ of strictly positive real numbers converging to zero. Then, by definition of the supremum, we associate a sequence of solution $(x_n(.))_{n\in \mathbb{N}}$:

$$\begin{array}{c} \forall \epsilon_n > 0, \exists T_n \geq 0, \exists x_n(.) \ \mid \ x_n(0) = x \text{ and } \forall t, x_n(t) \in K \\ \text{and } \overset{\infty\sup}{V}(x) - \epsilon_n \leq \int_0^{T_n} g(x_n(\tau))\, d\tau \leq \overset{\infty\sup}{V}(x) \end{array} \tag{44}$$

For all $t \in [0, T_n]$, $x_n(t)$ belongs to K, then $(x, \overset{\infty\sup}{V}(x) - \epsilon_n)$ belongs to $\text{Capt}_{(26)}(K \times \mathbb{R}^+, K, T_n)$, then to the union over T, $\text{Capt}_{(26)}(K \times \mathbb{R}^+, K)$. Then, similar to in the proof of Proposition 3.2, with $n \to \infty$, $(x, \overset{\infty\sup}{V}(x)) \in \text{Capt}_{(26)}(K \times \mathbb{R}^+, K)$.

Finally, $\overset{\infty\sup}{V}(x) = \sup_{(x,y)\in \text{Capt}_{(26)}(K\times\mathbb{R}^+,K)} y$.

□

4 Computing Viability Kernels

We begin with the simple viability problem {(4),(7)} with minimal consumption, then with the criterion of non-decreasing inter-temporal social welfare —the Brundtland criterion—, and then we shall check robustness. For the specific problem (23), the introduction of the auxiliary variable adds one dimension; even two in the autonomous case because time in System (23) is a variable in its own right through the discounting term $\exp(-\rho t)$. The computation must be done in state dimension greater than three, a task made possible by Bonneuil's (2006) viability algorithm. A variant of this algorithm (Bonneuil, forthcoming) is used to compute the boundary of the capture viability basin in the direction of high y without the knowledge of the whole viability kernel, which is very time-consuming. The principle involves two steps: firstly, a viable state is found for the auxiliary dynamic; secondly y is maximized for the same x. For each new attempt (x, y), a simulated annealing is performed to find one trajectory remaining in K and reaching the target C on time horizon T. For T infinite, an approximation and extrapolation of "reaching the target" is used, a task here made easy by the discounting term $\exp(-\rho t)$.

4.1 Minimal Consumption Viability

Figure 1 shows a set of attainable states from a given initial state x_0 and the viability kernel for constant minimal consumption $\underline{c}$ in the autonomous case. It contains all initial states from which an agent can navigate in capital and consumption, with the changes in consumption the agent can afford or she is compelled to make. At any time, the agent is able to safeguard minimal satisfaction, because, by definition, his or her state (k, c) remains in the set of constraints defined by Eq. (7). Within the viability kernel, any change in consumption is viable. The difficulty lies at the boundary of the viability kernel: on the boundary in the direction of high consumption, the only viable change in consumption is reduction at velocity $u^\flat := \inf_U u(t)$, leading to both reduction in consumption and capital, until reaching the boundary of the set of constraints itself, which occurs at $c = \underline{c}$, allowing the entrance into the interior of K again (semi-permeability property of Quincampoix, 1990). The boundary $c = \underline{c}$ of the viability kernel is also the boundary of the set of constraints, so that the control $u(t)$ is not necessarily unique (as indeed it is not in our Ramsey case study) .

In the autonomous case, with $f(k) := Ak^\alpha$ (with parameters A as technological level and α as capital share), when velocity is maintained at the minimum available $u^\flat$ (so that $c = c_0 + u^\flat t$), the corresponding boundary solution is a function $k = \kappa_{u^\flat}(c)$ at f, n, and δ fixed. The viability boundary in the direction of high c corresponds to the solution of:

$$d\kappa_{u^\flat} = \frac{1}{u^\flat}\left(f(\kappa_{u^\flat}) - c - (\delta + n)\kappa_{u^\flat}\right) dc \tag{45}$$

passing through $(f^{-1}(\underline{c}), \underline{c})$. The non autonomous case is less straightforward.

4.2 Optimal Inter-Temporal Consumption

The maximum of the integral of the net present utility is obtained by remaining on the boundary of the capture-viability basin of K in System {(23),(24)} in the direction of high y. With the specification $w(c) = \frac{c^{1-\sigma}}{1-\sigma}$ and $f(k) = Ak^\alpha$, Figure 2 presents states on the projection of this boundary onto the space (k, c, y). These states give maximal $\int_0^\infty w(c(\tau))e^{-\rho\tau}\, d\tau$ with (k, c) remaining in the set of constraints.

Figure 2 shows that intertemporal utility y can be maximized along other (viable) trajectories than the saddle path of Pontryagin maximization. One trajectory $(k(.), c(.), y(.))$ is represented: it leads from the initial state $x(0) = (k(0), c(0), 0, y(0) = y_0)$ to $(k(T), c(T), T, y(T) \approx 0)$. As $x(0)$ is on the boundary

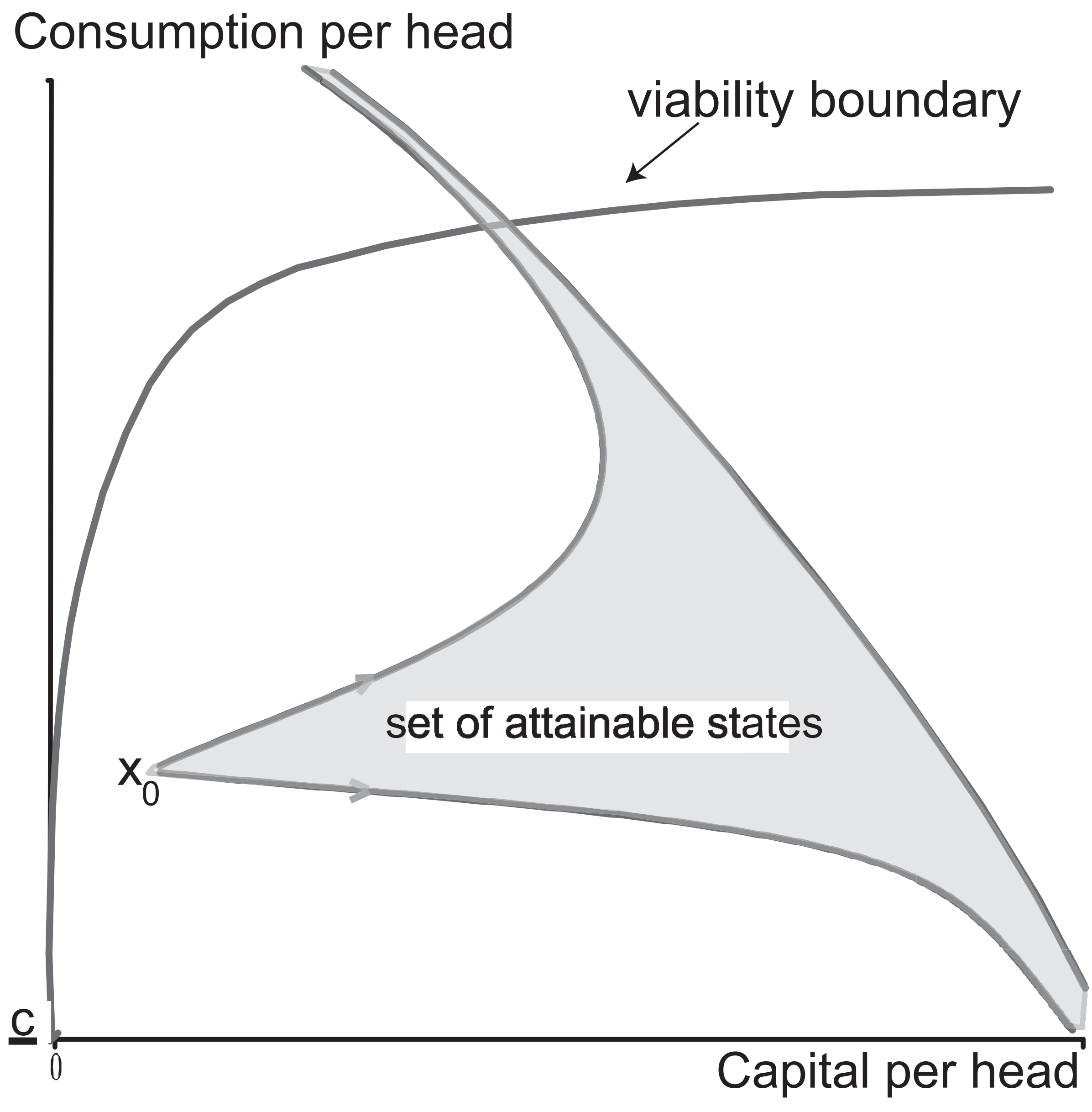

Figure 1: Viability kernel, delimited by the black line in the direction of high consumption, by $\underline{c}$ in the direction of low consumption, with an example of a set of attainable states. $f(k) = 0.04\sqrt{k}, n + \delta = 0.01$ (autonomous case).

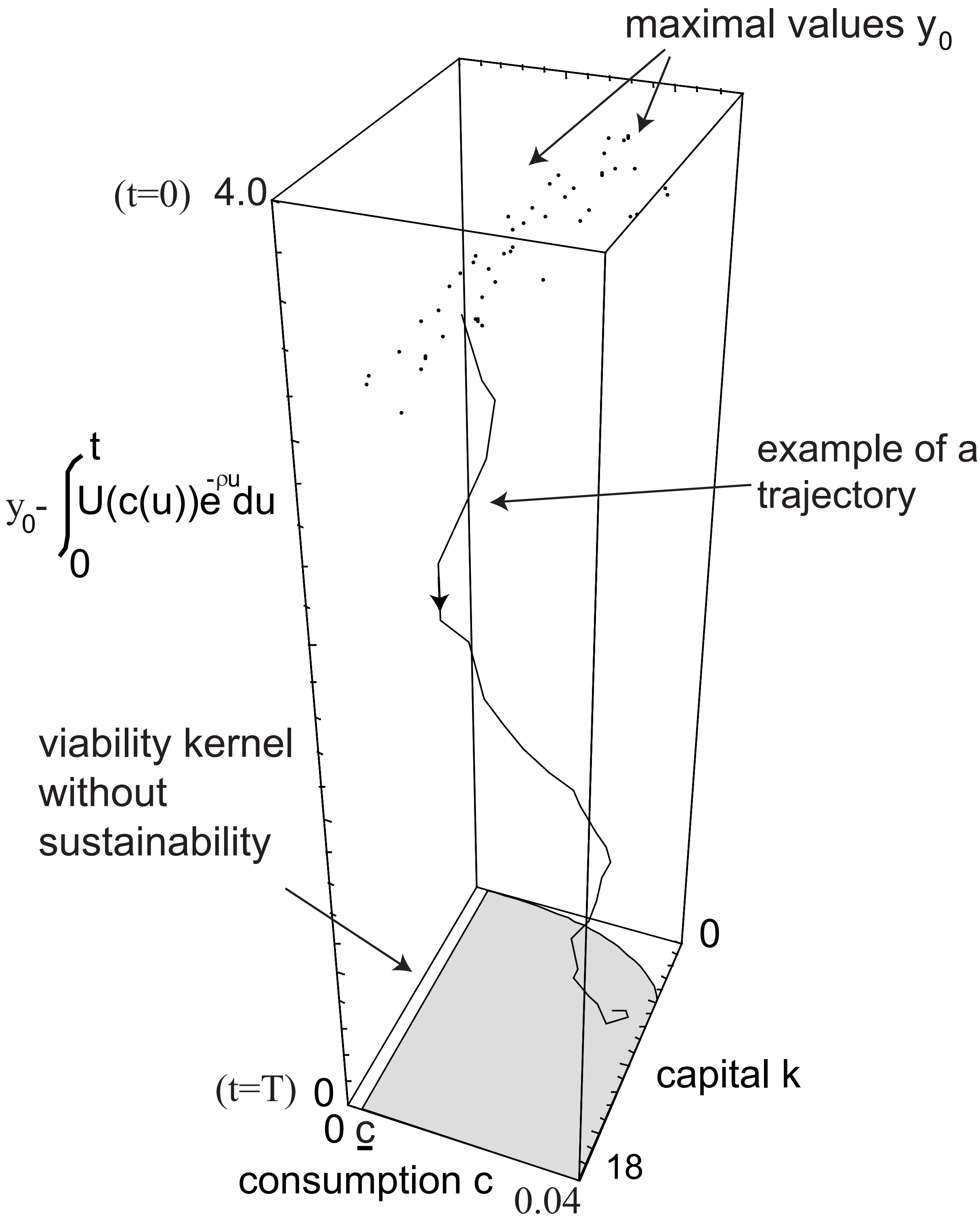

Figure 2: Maximal $\int_0^T w(c(\tau))e^{-\rho\tau}\,d\tau$ with $(k(.), c(.))$ remaining in the set of constraints. Case $f(k) = 0.04\sqrt{k}, n + \delta = 0.01, \rho = 0.05, \sigma = 0.5$.

of the viability-capture basin, y_0 is the maximal value attainable from $k(0), c(0)$ while satisfying the state and the control constraints all along the trajectory. Notably, no jump in consumption is allowed, as it is in the classical Ramsey framework. In the case of Figure 2, we recognize that $c(T) \approx c^*$, but there is a transitional phase during which $c(t)$ moves to c^* with its constraints U on consumption change. The slight "fluctuations" displayed by the trajectory on Figure 2 come about because of the numerical precision inherent in simulated annealing. They should not deflect attention from the efficiency with which Bonneuil's algorithm successfully identifies the correct trajectory and the correct boundary.

Obtaining the extremum of the integral criterion through viability has the advantage that constraints are specifically taken into account; they do not appear as a penalization in a static formulation using Lagrange multipliers. HJB requires properties of entering field (Soner condition). With viability, the delineation of the viability kernel allows the absence of entering field on the border of K. Also, both viability theory and algorithm support non-linearity naturally.

4.3 Brundtland Sustainability

In Section 2.2 we showed that Brundtland sustainability amounts to adding Eq. (10), which yields:

$$V(\theta) \geq \frac{w(c(\theta))}{\rho} \tag{46}$$

We introduce the auxiliary variable y, and with it, the time variable t involved in y if it is not present yet:

$$\left\{\begin{array}{rrcl} (i) & k'(t) & = & f(t,k(t)) - (n+\delta)k(t) - c(t) \\ (ii) & c'(t) & = & u(t) \\ (iii) & t' & = & 1 \\ (iv) & V'(t) & = & -w(c(t)) + \rho V(t) \\ (v) & y'(t) & = & -w(c(t))e^{-\rho t} \end{array}\right. \tag{47}$$

Satisfying the Brundtland condition amounts to having (k, c, t, V, y) remain in the closed set defined by the constraints {(24), (46)} until reaching the target $y = 0$ at the time horizon. The dynamics (47) is 5-dimensional. The numerical computation follows Bonneuil's (2006) algorithm. The robustness to parameters is addressed below.

The comparison in Figure 3 of the projection onto the plane $(k, c, 0)$ of the viability kernel with and without Brundtland sustainability highlights the necessity to reduce one's consumption with the hope of being Brundtland-sustainable. It also specifies the level of maximal consumption, when change in consumption is necessary (when the viability boundary is attained), and what this change u must be (such that there exists one trajectory remaining in K until a given time horizon T). Figure 1 presented the autonomous case, with a production function of the form $f(k) = Ak^\alpha$; Figure 3 presents the 3-dimensional non autonomous case with $f(t, k) = Ae^{\eta t}k^\alpha$ with a technological growth rate η.

Any Brundtland-viable state is viable without Brundtland sustainability, because the set of constraints with Brundtland is a subset of the set of constraints without Brundtland. This explains why the capture basin with Brundtland is included in the capture basin without it, a general result illustrated in Figure 3. We denote the maximum consumption, with $k' = 0$, in the autonomous system ($\eta = 0$) by $(\hat{k}^{**}, \hat{c}^{**})$, and define $k^{**}(t) = (A\exp(\eta t))^{1/(1-\alpha)}\hat{k}^{**}$ and $c^{**}(t) = (A\exp(\eta t))^{1/(1-\alpha)}\hat{c}^{**}$. The fact that the boundary of the viability kernel with Brundtland increases with k is due to positive technological progress, which enables trajectories starting from certain states with a high $k(0)$ and a $c(0) > c^{**}(0)$ to attain a $c(t) \leq c^{**}(t)$. Then, the system has the possibility of sliding over time at the same $k(t)$, and then of being Brundtland-viable. The fact that the boundary of the viability kernel without Brundtland increases with k is not as simple as in the autonomous case (Eq. (45)), hence the need for an algorithm.

4.4 Optimal Inter-Temporal Consumption with Brundtland Sustainability

Similar to our computation of the viable-optimal solution in the Ramsey model, the viable-optimal solution in the Ramsey model with Brundtland sustainability is obtained on the boundary of the viability kernel in the direction of high y of the 5-dimensional auxiliary system (47). The constraint of inequality (46) was added. The 3D-projection onto the space (k, c, y) is represented on

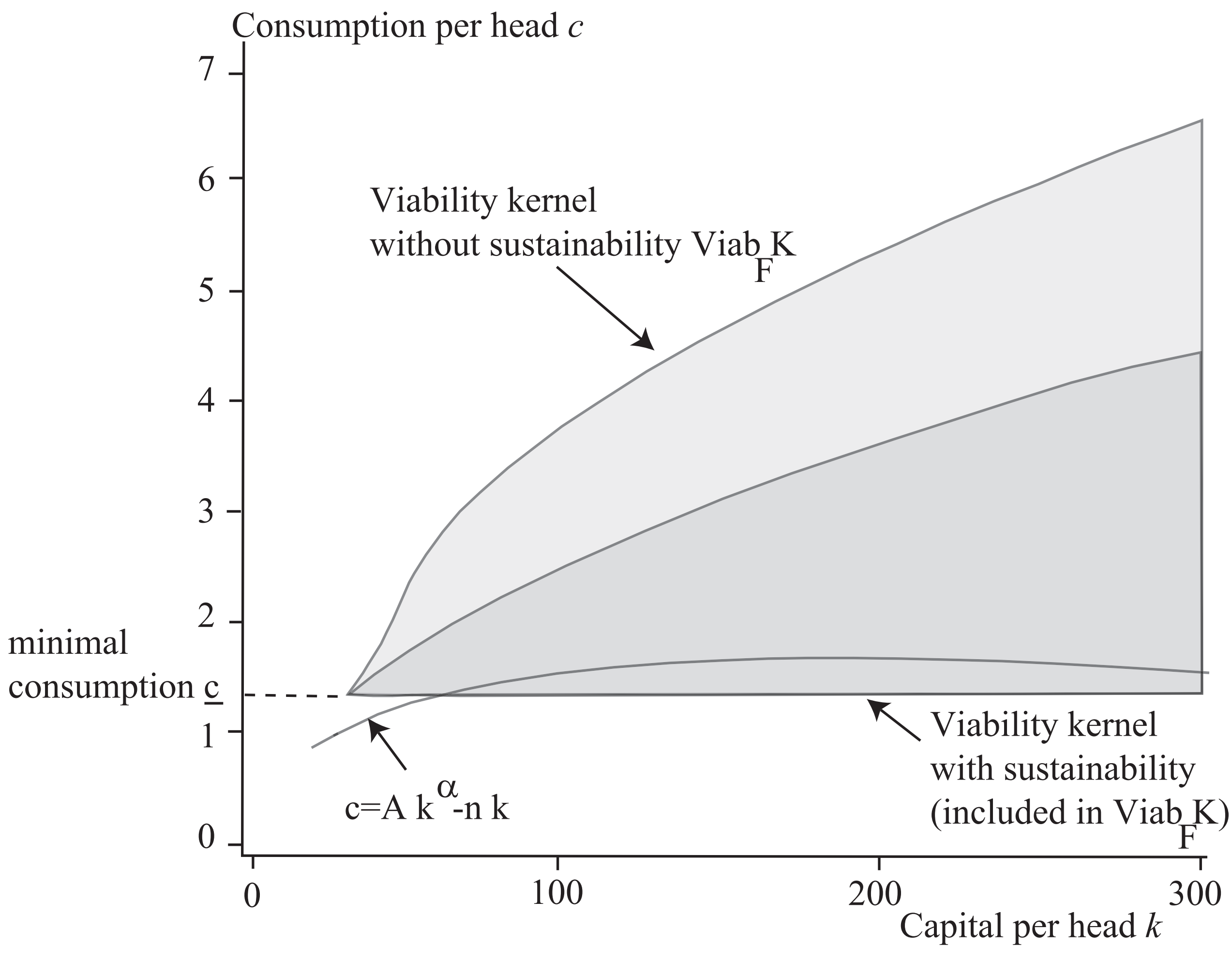

Figure 3: Comparison of the projections onto the plane $(k, c, 0)$ of the viability kernel with and without Brundtland sustainability (non autonomous case). Case $f(k) = 0.21e^{0.005t}k^{0.55}$, $n + \delta = 0.011$, $\sigma = 0.5$, $\rho = 0.02$, $\underline{c} = 1.32$, $u^\flat = -0.034$, $T = 200$.

Figure 4.

4.5 Robustness

Using the Bonneuil (2006) algorithm, we computed viability kernels with and without the Brundtland criterion for 200 different sets of parameters drawn at random. The parameters are:

- the elasticity σ in the iso-elastic utility function $w(c) = \frac{c^{1-\sigma}}{1-\sigma}$; the range of variation of σ was taken as [0.2, 0.8];
- the technological progress level A (ranging in $[0.01, 0.5]$), the technological progress rate η (ranging in $[0.0, 0.03]$), and the capital share α (ranging in $[0.3, 0.7]$) in the production function $A \exp(\eta t) k^{\alpha}$;
- the consumption threshold $\underline{c}$ (positive);
- the discounting factor ρ, which to satisfy the requirement of integral convergence for $T = \infty$, must be greater than $(1-\sigma)\eta$, so ρ was drawn randomly from $[(1-\sigma)\eta, 0.05]$);
- the population growth rate n (in $[-0.02, 0.02]$);
- the set of controls, with $u^{\flat}$ varying in $[-0.01, 0]$ and $u^{\sharp}$ in $[0, 0.01]$;
- T was set large enough so that all trajectories starting from any k in the $(k, c, 0)$ plane have the time to leave the set of constraints.

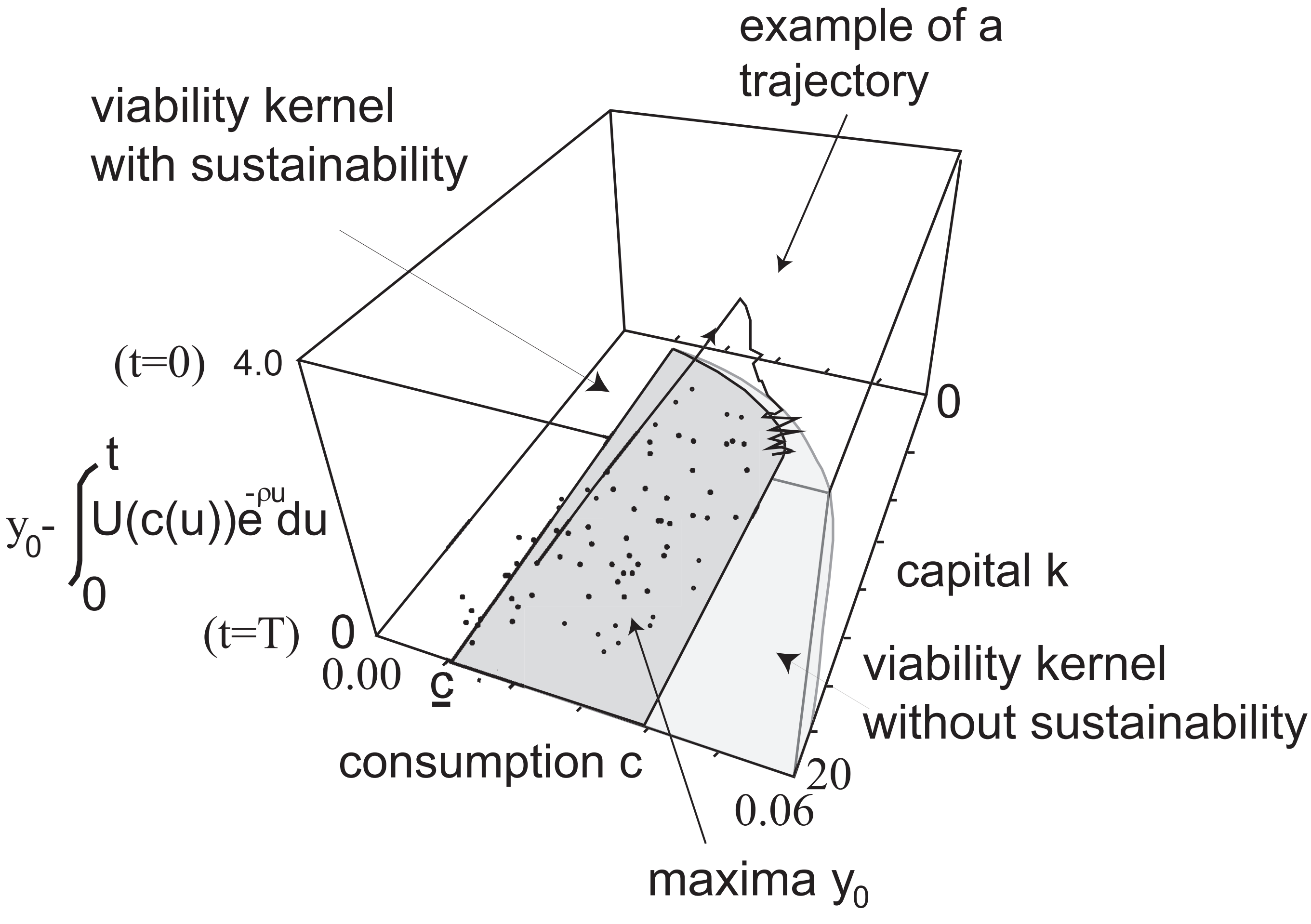

Figure 4: Maximal $\int_0^T w(c(\tau))e^{-\rho\tau}\,d\tau$ with $(k(.), c(.), t(.), V(.))$ remaining in the set of constraints, and particular trajectory leading to maximal $\int_0^T w(c(\tau))e^{-\rho\tau}\,d\tau$ with $(k(.), c(.), t(.), V(.))$ remaining in the set of constraints. Case $f(k) = 0.04\sqrt{k}, n + \delta = 0.01, \rho = 0.05, \sigma = 0.5$.

The regression of the volume of the viability kernel suffers from a selection issue, because we measure the effects of co-variates only when we observe the non emptiness of the viability kernel. This difficulty is solved by the two-stage Heckman procedure (Heckman, 1979; Wooldridge, 2002; Cameron and Trivedi, 2005). We recall that the dependency of the point $k^{**}(t), c^{**}(t)$ of maximum consumption with $k'(t) = 0$ on parameters is:

$$k^{**}(t) = (\frac{\alpha A e^{\eta t}}{n+\delta})^{\frac{1}{1-\alpha}} \tag{48}$$

and

$$c^{**}(t) = f(k^{**}(t)) - (n+\delta)k^{**}(t) = c^{**}(0)e^{\frac{\eta t}{1-\alpha}} \tag{49}$$

For $0 \in U$, the viability kernel without sustainability is trivially empty for $\underline{c} > c^{**}(T)$, and trivially non empty for $\underline{c} \leq c^{**}(T)$. The identification of emptiness with the value of $\underline{c}$ with respect to the state of maximum consumption $c^{**}(t)$ for which $k' = 0$ means that the inverse Mills ratio, an additional term introduced in the regression of the volume to correct the bias resulting from using a non-randomly selected sample (Heckman, 1979) –here the condition that the volume is non empty–, is automatically set to zero in the regression.

Similarly, for the viability kernel with sustainability, all states over $c^{**}(T)$ are passed through by trajectories which either leave K in finite time or require $c' < 0$ at least once, thus, from Proposition 2.1, these states are not Brundtland-viable. A state sliding along $(k^{**}(t), c^{**}(t))$, which is allowed when $0 \in U$, has a non decreasing net present value and is Brundtland sustainable. Again, the

Heckmann procedure sets the inverse Mills ratio automatically to zero, because non emptiness is equivalent to $\underline{c} \leq c^{**}(T)$.

For the volume of the viability kernel without sustainability, Table 1 confirms what can be guessed from Figure 1 namely that the higher the consumption threshold $\underline{c}$, the smaller the viability kernel; and the higher the production coefficient A, the larger the viability kernel; the same is true for the technology growth rate η which accompanies A. Conversely, the population growth rate n, which appears in the expression of c^{**} for example with an effect contrary to that of A, has a decreasing effect on the volume of the viability kernel: population growth, in this Malthusian model, reduces wealth per head. The lower the minimal control $u^\flat$, the greater the scope for reducing one's standard of living and hence the larger the viability kernel.

For the volume of the viability kernel with sustainability, Table 2 shows the effect of relevant variables, notably the positive effect of $k^{**}(0)$ or $c^{**}(0)$ which is strongly correlated with $k^{**}(0)$, and the additional positive effect of the technological growth rate η. Proposition 2.1 implies that the value of the minimal change $u^\flat < 0$ in consumption plays no role on the volume of the viability kernel with sustainability, as the simulation confirms, and that non emptiness requires that the value of the maximal change $u^\sharp$ be non negative: at some time, the system must have a non decreasing consumption to prevent V from decreasing. Neither the discount rate ρ nor elasticity σ play any significant role. These two parameters slightly distort the set of equilibria and the viability kernel, but not sufficiently for this to be significant with the method of computing kernels by drawing points at random. The variation of the viability kernel with other parameters (technological level A, population growth rate n, capital share α) passes through the dependency on $c^{**}(t)$.

Table 1: Regression (two-step Heckman procedure) of the volume of the viability kernel *without* sustainability for $\underline{c} \leq c^{**}(T)$ (all variables standardized in [0,1]).

Variable	Estimate	Standard deviation
Intercept	0.04	0.03
minimal change $u^\flat$	-0.42*	0.02
maximal change $u^\sharp$	-0.01	0.02
technology level A	0.40*	0.03
technology growth rate η	0.17*	0.03
capital share α	0.30*	0.03
population growth rate n	-0.08*	0.03
minimal consumption $\underline{c}$	-73.44*	5.34
inverse Mills ratio	0.00	.

*: significant at the 5% level N=200, R=0.81

Table 2: Regression (Two-step Heckman procedure) of the volume of the viability kernel *with* sustainability for $\underline{c} \leq c^{**}(T)$ (all variables standardized in [0,1]).

Variable	Estimate	Standard deviation
Intercept	0.07	0.03
technology growth rate η	0.26*	0.03
maximal consumption c^{**} where $k' = 0$	25.15*	1.58
minimal consumption $\underline{c}$	-62.80*	4.23
discount rate ρ	-0.02	0.03
elasticity σ	0.02	0.03
inverse Mills ratio	0.00	.

*: significant at the 5% level N=200, R=0.74

Note: $u^\flat$ and $u^\sharp$ contained in c^{**}

5 Conclusion

Re-visiting the Ramsey model of neoclassical optimal growth, we enriched the theory by completing the usual concepts of optimal path and of equilibria with that of viable state. This allowed us to diversify the classical study of the sole optimal solution converging to the steady state: instead of looking at where the system goes, turned the question on its head: to satisfy given constraints, what are the initial states from which this is possible, and what are the changes in consumption that make this possible? By proceeding thus, we no longer needed to add an ad-hoc long-term outcome term to the traditional inter-temporal optimization of utility as in Chichilnisky (1993).

We delineated the viability kernel –the largest set of such viable states– in the autonomous and in the non autonomous cases. We went on to consider the constraint reflecting economic sustainability in the sense of the Brundtland commission, before situating optimality in the Pontryagin sense, Brundtland sustainability, and viability in relation with each other. We compared viability kernels with or without sustainability and showed by how much consumption must be reduced to ensure sustainability. Being outside the viability-capture basin requires an abrupt reduction in consumption to a Brundtland viable state, followed by the implementation of viable policies.

The viable-optimal solution in the sense of inter-temporal consumption is optimal among the solutions continuously satisfying the constraints. It is no longer obtained through Pontryagin, and we showed that it is obtained on the capture-viability boundary of an auxiliary system. We solved the similar problem augmented for Brundtland sustainability. We then successfully combined viability, optimality, and sustainability.

On the applied side, we revealed significant technological, demographic, consumption ($\underline{c}$), and decision-making ($u^\flat$) determinants of viability kernels. As expected in the Ramsey model which is Malthusian in spirit (population growth always has a negative effect on wealth), population growth reduces the viability kernel, favoring the movement of the state toward poverty ($k = 0$), while technological progress always increases the viability kernel, giving more room to manoeuvre against impoverishment. We showed that technological progress works against population growth to favor the possibility for a given state of being viable or viable-sustainable.

We suggest that this new concept of viability and optimality-viability, which reveals the potential of economic models, in thi scase the simple Ramsey model, to account for the presence of constraints and for the intrinsic diversity and unpredictability in the resulting behaviors and destinies of the system.

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