Computing Markov-Perfect Optimal Policies in Business-Cycle Models[*]

Richard Dennis[†]
University of Glasgow

Tatiana Kirsanova
University of Glasgow

November 2014

Abstract

Time-inconsistency is an essential feature of many policy problems (Kydland and Prescott, 1977). This paper presents and compares three methods for computing Markov-perfect optimal policies in stochastic nonlinear business cycle models. The methods considered include value function iteration, generalized Euler-equations, and parameterized shadow prices. In the context of a business cycle model in which a fiscal authority chooses government spending and income taxation optimally, while lacking the ability to commit, we show that the solutions obtained using value function iteration and generalized Euler equations are somewhat more accurate than that obtained using parameterized shadow prices. Among these three methods, we show that value function iteration can be applied easily, even to environments that include a risk-sensitive fiscal authority and/or inequality constraints on government spending. We show that the risk-sensitive fiscal authority lowers government spending and income-taxation, reducing the disincentive households face to accumulate wealth.

Keywords: *Markov-Perfect Policy, Time-Consistency, Fiscal Policy*

JEL Classification: E62, C61, C63.

1 Introduction

Governments are confronted routinely with important policy decisions, not least with decisions regarding spending and taxation (i.e., fiscal policy) and about how to set a policy interest rate (i.e., monetary policy). Following Kydland and Prescott (1977) it is now well-known that, when private agents are forward-looking and rational, optimal policies are (invariably) not time-consistent and that time-consistent policies are (invariably) not optimal. Absent a commitment technology, attention focuses naturally on policies that are time-consistent or, more specifically, on equilibria that are Markov-perfect and this focus has led to a large literature on "discretionary" policymaking. While there are notable exceptions, the vast bulk of this literature computes equilibrium using what is essentially linear-quadratic dynamic programming with the policy problem approximated to have the required linear constraints and quadratic objective. Although some policy problems can conceivably be well-approximated by linear constraints and second-order accurate welfare approximations, such as policy problems for which the steady state is efficient, and for other problems, such as those where the model has many state variables, only linear-quadratic methods may be feasible, for many interesting problems, such as problems involving distortionary taxes and/or imperfect competition, first-order accuracy cannot be obtained through linear-quadratic methods. Moreover, some model characteristics, like occasionally binding constraints, call out for nonlinear solution methods even when the policy objective is quadratic (Adam and Billi, 2007; Nakata, 2012). Fortunately, it is becoming increasingly feasible to analyze discretionary policymaking without turning to the linear-quadratic toolkit.

In this paper we present and compare three strategies for computing Markov-perfect optimal policies in nonlinear stochastic business cycle models. These strategies apply to economies populated by a large number of atomistic private agents and by a benevolent government, tasked with conducting policy in order to maximize the welfare of the representative household. The solution strategies include value function iteration, policy function iteration on generalized Euler equations, and parameterized shadow prices. To illustrate the three strategies we apply them to the canonical dynamic model of fiscal policy taken from Klein, Krusell, and Ríos-Rull (2008), augmented to include an transitory aggregate technology shock and extended to allow for capital depreciation (as per Ambler and Pelgrin, 2010). In this model, government spending provides households with utility and the fiscal authority's problem is to choose government spending optimally, subject to a balanced-budget constraint, while lacking both a commitment technology

and the ability to impose lump-sum taxes. This balanced-budget fiscal policy model provides the ideal environment in which to illustrate the methods because this model, or closely related models, have been studied by Stockman (2001), Benhabib and Rustichini (1997), Klein and Ríos-Rull (2003), Ortigueira (2006), Klein, Krusell, and Ríos-Rull (2008), and Ambler and Pelgrin (2010). However, with the exception of Ambler and Pelgrin (2010), each of these studies has looked only at deterministic specifications and/or focused only on the model's steady state behavior.[1]

For the canonical fiscal policy model we find that value function iteration and the generalized Euler equations solution method, both employing Chebyshev polynomials for function approximation and Gauss-Hermite quadrature for integration, perform well and are notably more accurate in terms of Euler-equation errors than the parameterized shadow prices solution approach, which is based on parameterized expectations and uses Monte Carlo integration. We extend the canonical model to allow for a risk-sensitive fiscal authority and for an inequality constraint on government spending as a share of output and show how the equilibrium of these models can be computed easily, and relatively accurately, using the value function iteration solution method. We show that the risk-sensitive fiscal authority cuts government spending in order to reduce income taxation and thereby mitigate the disincentive on household capital accumulation.

Our paper is related to several others. In particular, we take our main model— a balanced-budget fiscal policy problem— from Ambler and Pelgrin (2010), who show how a method of parameterized expectations can be used to compute its time-consistent equilibrium. Our use of Chebyshev polynomials for approximating functions is shared by a number of studies, including Ortigueira (2006), Niemann, Pichler, and Sorger (2008, 2009), Ortigueira and Pereira (2009), and Anderson, Kim, and Yun (2010). Our paper is also related to the important literature on discretionary policymaking in linear-quadratic models, and to the computational strategies of Kydland and Prescott (1977), Oudiz and Sachs (1985), Currie and Levine (1985, 1993), Backus and Driffill (1986), Söderlind (1999), and Dennis (2007).

The remainder of the paper is organized as follows. Section 2 describes a simple balanced-budget business cycle model of fiscal policy. This model is interesting in its own right and usefully serves as a vehicle for illustrating and comparing the various solution strategies. Section 3 presents the three solution strategies. Section 4 applies the various solution strategies to solve

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[1] Benhabib and Rustichini (1997), Domínguez (2007), and Ortigueira and Pereira (2009) consider related models in which the fiscal authority's balanced-budget restriction is relaxed. But these studies, too, focus on deterministic models.

the fiscal policy model for its Markov-perfect equilibrium. Section 5 shows how features such as occasionally binding constraints and risk-sensitivity can be accommodated. Section 6 discusses computation times and the application of these methods to larger models. Section 7 offers concluding remarks.

2 The model

We consider a production economy populated by a unit-mass of identical atomistic households, a unit-mass of identical atomistic firms, and a fiscal authority. Firms rent capital and hire labor from households and use these inputs to produce goods that are sold to households and the fiscal authority. Goods sold to the fiscal authority are transformed costlessly into a government consumption good while those sold to households are either consumed or used to augment the capital stock. The fiscal authority taxes household income, using the revenue to finance the provision of the government consumption good. Markets are assumed to be perfectly competitive. This model follows Ambler and Pelgrin (2010) and extends Klein, Krusell, and Ríos-Rull (2008).

2.1 Households

Households own the capital stock. They receive income by renting their capital and supplying their labor to firms at prices r_t and w_t, respectively. After paying income tax, households use their remaining income to purchase goods, which they use to offset capital-depreciation, to invest in their capital stock, and to consume. The representative household's lifetime utility function is described by

$$\mathrm{E} \sum_{t=0}^{\infty} \beta^t u\left(c_t, G_t\right), \tag{1}$$

where $\beta \in (0, 1)$ is the discount factor, c_t denotes private consumption, G_t denotes government consumption goods, and the momentary utility function $u\left(c_t, G_t\right)$ is assumed to be strictly increasing, strictly concave, twice continuously differentiable, and to satisfy the Inada (1963) conditions. The capital owned by the representative household evolves over time according to

$$k_{t+1} + c_t = k_t + (1 - \tau_t)\left[(r_t - \delta) k_t + w_t\right], \tag{2}$$

where $\delta \in (0, 1)$ is the depreciation rate, $\tau_t > 0$ is the tax rate applied to household income (with a tax-allowance for capital-depreciation), and k_t is the household's stock of capital as of the beginning of period t. Households maximize their expected lifetime utility, equation (1), subject

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to their flow-budget constraint, equation (2), taking prices, taxes, and government consumption goods as given.

2.2 Firms

Our stand-in aggregate firm employs capital and labor to produce output according to the neo-classical production technology

$$Y_t = e^{z_t} F(K_t, 1) = e^{z_t} F(K_t), \tag{3}$$

where Y_t represents aggregate output, K_t denotes the aggregate capital stock as of the beginning of period t, and z_t is an aggregate technology shock that obeys the stochastic process

$$z_{t+1} = \rho z_t + \epsilon_{t+1}, \tag{4}$$

where $\rho \in (0, 1)$ and $\epsilon_t \sim i.i.d. \left[0, \sigma_\epsilon^2\right]$.

Markets for capital and labor are perfectly competitive and clear at the prices

$$r_t = e^{z_t} F_K(K_t), \tag{5}$$

$$w_t = e^{z_t} F(K_t) - e^{z_t} F_K(K_t) K_t, \tag{6}$$

respectively, with the stand-in firm making zero-profits in equilibrium.

2.3 Fiscal authority

The fiscal authority cannot impose lump-sum taxes, but receives revenue by taxing household income at marginal rate τ_t. These tax revenues are used to purchase goods that are costlessly transformed into government consumption goods and provided to households at zero unit-cost. The fiscal authority has no outstanding liabilities and cannot issue bonds. As a consequence, the fiscal authority's decisions about taxation and the provision of the public good, decisions made to maximize the welfare of the representative household, are constrained by the balanced-budget condition

$$G_t = \tau_t \left[(r_t - \delta) K_t + w_t\right], \tag{7}$$

where G_t denotes aggregate government consumption.

2.4 Information, timing, and aggregation

With the current realization for the aggregate technology given by z_t, we denote the history of realizations for aggregate technology up to and including period t by $z^t = \{z_i\}_{i=0}^t$. Similarly, using $\mathbf{x}_t = \begin{bmatrix} z_t & k_t & K_t \end{bmatrix}'$ to denote the economy's state at the beginning of period t, we assume that at the beginning of period t all agents are endowed with the information set given by the history $\mathbf{x}^t$. After entering period t, and having observed $\mathbf{x}^t$, the fiscal authority, firms, and households make their decisions simultaneously. This timing protocol is also considered by Cohen and Michel (1988), Ortigueira (2006), and Ambler and Pelgrin (2010) and is relatively common in the literature on optimal fiscal policy.[2] Our assumptions that all households and all firms are identical and that they are of unit-mass implies that $K_t = k_t$ and $C_t = c_t$ in aggregate.

3 Solving for Markov-perfect optimal policy

In this section we present three ways to solve this fiscal-policy model for a Markov-perfect equilibrium. More generally, however, this model is quite representative of the fiscal- and monetary-policy problems that we are interested in, and the solution methods presented here can be applied quite broadly to these problems. As we shall see, the value function iteration method that we describe can also be applied with relative ease to more sophisticated policy problems, ones involving features such as Epstein-Zin preferences, risk-sensitive preferences, and/or occasionally binding constraints.

3.1 Value function iteration

The problem facing the representative household can be represented by the Bellman equation

$$v\left(z_t, k_t, K_t\right) = \max_{c_t, k_{t+1}} \left[u\left(c_t, G_t\right) + \beta \mathrm{E}_t \left[v\left(z_{t+1}, k_{t+1}, K_{t+1}\right) \right] \right], \tag{8}$$

with the constraints given by the laws-of-motion for household-level and aggregate capital, respectively,

$$k_{t+1} = k_t + \left(1 - \frac{G_t}{e^{z_t} F\left(K_t\right) - \delta K_t} \right) \left[\left(e^{z_t} F_K\left(K_t\right) - \delta\right) k_t + e^{z_t} F\left(K_t\right) - e^{z_t} F_K\left(K_t\right) K_t \right] - c_t \tag{9}$$

$$K_{t+1} = \left(1 - \delta\right) K_t + e^{z_t} F\left(K_t\right) - C_t - G_t. \tag{10}$$

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[2] In the monetary policy literature it is more common to assume that the government has a first-mover advantage within the period. See for example Clarida, Galí, and Gertler (1999) and Woodford (2003). Martin (forthcoming) uses a model related to Klein, Krusell, and Ríos-Rull (2008) to study the effect on welfare of different timing protocols.

taking G_t and C_t as given, and with the initial conditions $z_t > 0$ and $k_t = K_t > 0$, known. Notice that equations (5) and (6) have been used so that equation (9) does not contain prices. Combining the first-order condition with respect to consumption, c_t, and the Benveniste and Scheinkman (1979) condition, and aggregating across identical households gives the consumption-Euler equation

$$u_c\left(C_t, G_t\right) = \beta \mathrm{E}_t\left[u_c\left(C_{t+1}, G_{t+1}\right)\left(1 + \left(1 - \frac{G_{t+1}}{e^{z_{t+1}}F\left(K_{t+1}\right) - \delta K_{t+1}}\right)\left(e^{z_{t+1}}F_K\left(K_{t+1}\right) - \delta\right)\right)\right]. \tag{11}$$

The fiscal authority's problem is described by the Bellman equation

$$V\left(z_t, K_t\right) = \max_{G_t, K_{t+1}}\left[u\left(C_t, G_t\right) + \beta \mathrm{E}_t\left[V\left(z_{t+1}, K_{t+1}\right)\right]\right], \tag{12}$$

with the constraints given by equation (10) with the initial conditions $z_t > 0$ and $K_t > 0$, known.

3.2   Equilibrium

A Markov-perfect Nash equilibrium for this model is a collection of household decision rules, $\{c\left(z_t, k_t, K_t\right), k\left(z_t, k_t, K_t\right)\}$, a collection of aggregate decision rules, $\{C(z_t, K_t), G(z_t, K_t), K(z_t, K_t)\}$, and a collection of value functions, $\{v(z_t, k_t, K_t), V(z_t, K_t)\}$, such that

1. The collection $\{v(z_t, k_t, K_t), c(z_t, k_t, K_t), k(z_t, k_t, K_t)\}$ solves the household's decision problem described by the Bellman equation, equation (8), and the constraints, equations (9) and (10).

2. The collection $\{V(z_t, K_t), C(z_t, K_t), K(z_t, K_t), G(z_t, K_t)\}$ solves the fiscal authority's decision problem described by the Bellman equation, equation (12), and the constraint, equation (10).

3. $k_t = K_t$, $C(z_t, K_t) = c(z_t, K_t, K_t)$, and $k(z_t, K_t, K_t) = K(z_t, K_t)$.

As we describe later, to solve numerically the fiscal authority's problem we conjecture functions for aggregate consumption, $C\left(z_t, K_t\right)$, aggregate government spending, $G\left(z_t, K_t\right)$, and the value function, $V\left(z_t, K_t\right)$. Based on these conjectured functions and the law-of-motion for aggregate capital, we solve the fiscal authority's problem using a hill-climber, obtaining updates for $G\left(z_t, K_t\right)$ and $V\left(z_t, K_t\right)$, the former of which, when combined with equation (11) delivers an update for $C\left(z_t, K_t\right)$. Iterating to convergence we arrive at a Markov perfect equilibrium for this policy problem.

3.3 Generalized Euler equations

To solve the model for a Markov-perfect equilibrium using generalized Euler equations, we return to the fiscal authority's decision problem, which is described by the Bellman equation

$$V(z_t, K_t) = \max_{G_t, K_{t+1}} [u(C_t, G_t) + \beta \mathrm{E}_t [V(z_{t+1}, K_{t+1})]], \tag{13}$$

and the constraint

$$K_{t+1} = (1 - \delta) K_t + e^{z_t} F(K_t) - C_t - G_t. \tag{14}$$

The first-order condition with respect to G_t gives

$$u_G(C_t, G_t) = \beta \mathrm{E}_t [V_K(z_{t+1}, K_{t+1})], \tag{15}$$

while the Benveniste-Scheinkman condition yields

$$\begin{aligned} V_K(z_t, K_t) &= u_C(C_t, G_t) C_K(z_t, K_t) + u_G(C_t, G_t) G_K(z_t, K_t) \\ &+ \beta \mathrm{E}_t [V_K(z_{t+1}, K_{t+1})] (1 - \delta + e^{z_t} F_K(K_t) - C_K(z_t, K_t) - G_K(z_t, K_t)), \end{aligned} \tag{16}$$

and together equations (15) and (16) imply the generalized Euler equation

$$\begin{aligned} u_G(C_t, G_t) &= \beta \mathrm{E}_t [(u_C(C_{t+1}, G_{t+1}) - u_G(C_{t+1}, G_{t+1})) C_K(z_{t+1}, K_{t+1})] \\ &+ \beta \mathrm{E}_t [u_G(C_{t+1}, G_{t+1}) (1 - \delta + e^{z_{t+1}} F_K(K_{t+1}))], \end{aligned} \tag{17}$$

which is "generalized" because it contains both the level and the derivative of $C(z_{t+1}, K_{t+1})$.

To solve for a Markov-perfect equilibrium one needs to solve the system consisting of equations (14), (17), and (11) from the household's problem, with the process for the aggregate technology shock given by equation (4). To solve this system we conjecture functions for aggregate consumption, $C(z_t, K_t)$, aggregate government spending, $G(z_t, K_t)$, and iterate on the system until a fixed-point is reached.

3.4 Parameterized shadow prices

To solve the model for a time-consistent equilibrium using parameterized shadow prices (Ambler and Pelgrin, 2010), we formulate the household's problem in terms of a Lagrangian

$$\mathrm{E} \sum_{t=0}^{\infty} \beta^t \left[\begin{array}{c} u(c_t, G_t) \\ + \phi_t \left(k_t + \left(1 - \frac{G_t}{e^{z_t} F(K_t) - \delta K_t}\right) [(e^{z_t} F_K(K_t) - \delta) k_t + e^{z_t} F(K_t) - e^{z_t} F_K(K_t) K_t] - c_t - k_{t+1} \right) \end{array} \right], \tag{18}$$

and derive the first-order conditions with respect to c_t, k_{t+1}, and ϕ_t, which after aggregating across identical households (implying $\Phi_t = \phi_t$) are, respectively,

$$u_C(C_t, G_t) = \Phi_t,$$

$$\Phi_t = \beta E_t \left[\left(1 + \left(1 - \frac{G_{t+1}}{e^{z_{t+1}} F(K_{t+1}) - \delta K_{t+1}} \right) (e^{z_{t+1}} F_K(K_{t+1}) - \delta) \right) \Phi_{t+1} \right],$$

$$K_{t+1} = K_t + \left(1 - \frac{G_t}{e^{z_t} F(K_t) - \delta K_t} \right) \left[(e^{z_t} F_K(K_t) - \delta) k_t + e^{z_t} F(K_t) - e^{z_t} F_K(K_t) K_t \right] - C_t.$$

Now recognizing that the shadow price Φ_t will be a function of only z_t and K_t in a simultaneous-move time-consistent equilibrium, the fiscal authority's problem is formulated using the Lagrangian

$$E \sum_{t=0}^{\infty} \beta^t \left[\begin{array}{c} u(C_t, G_t) + \Xi_t (u_C(C_t, G_t) - \Phi(z_t, K_t)) \\ + \Psi_t ((1-\delta) K_t + e^{z_t} F(K_t) - C_t - G_t - K_{t+1}) \end{array} \right]. \tag{19}$$

The first-order conditions from the fiscal authority's decision problem yield

$$u_C(C_t, G_t) = \Psi_t - u_{CC}(C_t, G_t) \Xi_t, \tag{20}$$

$$u_G(C_t, G_t) = \Psi_t - u_{CG}(C_t, G_t) \Xi_t, \tag{21}$$

$$\Psi_t = \beta E_t \left[(1 - \delta + e^{z_{t+1}} F_K(K_{t+1})) \Psi_{t+1} - \Phi_K(z_{t+1}, K_{t+1}) \Xi_{t+1} \right], \tag{22}$$

$$u_C(C_t, G_t) = \beta E_t \left[\left(1 + \left(1 - \frac{G_{t+1}}{e^{z_{t+1}} F(K_{t+1}) - \delta K_{t+1}} \right) (e^{z_{t+1}} F_K(K_{t+1}) - \delta) \right) u_C(C_{t+1}, G_{t+1}) \right] \tag{23}$$

$$K_{t+1} = (1-\delta) K_t + e^{z_t} F(K_t) - C_t - G_t. \tag{24}$$

The time-consistent equilibrium is now obtained by solving equations (20)—(24), and (4) using parameterized expectations (Marcet and Lorenzoni, 1999), to approximate the expectation terms in equations (22) and (23).

4   Solving the model

In this section we solve the model presented in Section 2 using the three methods discussed in Section 3.   For this exercise, we assume that the representative household's momentary utility function is of the additively separable form

$$u(c_t, G_t) = \frac{c_t^{1-\sigma} - 1}{1 - \sigma} + \mu \frac{G_t^{1-\eta} - 1}{1 - \eta}, \tag{25}$$

where $\{\sigma, \eta, \mu\} > 0$, and that the production function is

$$Y_t = e^{z_t} K_t^{\alpha}, \tag{26}$$

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where $\alpha \in (0, 1)$.

Our parameterization of the model follows Ambler and Pelgrin (2010) and is summarized in Table 1.[3]

Table 1: Benchmark Parameterization		
Parameter	Value	Interpretation
β	0.987	Household discount factor
σ	1.000	Utility curvature of private consumption
μ	0.300	Utility weight on government services
η	1.000	Utility curvature of government services
α	0.300	Capital-share of output
δ	0.050	Depreciation rate
ρ	0.950	Persistence of technology shock
σ_ϵ	0.030	Standard deviation of technology shock

To compute the Markov-perfect equilibrium using value function iteration (VFI) and generalized Euler equations (GEE) we use Chebyshev polynomials to approximate the conjectured functions. In the case of GEE, the procedure requires introducing polynomials to approximate the decision rules for consumption, $C(z_t, K_t)$, and government spending, $G(z_t, K_t)$, while VFI requires, in addition, a polynomial to approximate the value function, $V(z_t, K_t)$. For a generic function, $X(z_t, K_t)$, these polynomial approximations take the form

$$X(z_t, K_t) \simeq \sum_{j=1}^{n^z} \sum_{i=0}^{n^k} w_{ij} \Gamma_i(K_t) \Gamma_j(z_t), \qquad (27)$$

where $\Gamma_i(K_t)$ represents the i'th term of the Chebyshev polynomial in aggregate capital, $\Gamma_j(z_t)$ represents the j'th term of the Chebyshev polynomial in aggregate technology, n^k and n^z represent the orders of the Chebyshev polynomials, and w_{ij} represents the Chebyshev weights.

For VFI, at each node for capital and technology we use a Newton-based hill climber to solve

$$V(z_t, K_t) = \max_{G_t, K_{t+1}} \left[\frac{C(z_t, K_t, G_t)^{1-\sigma} - 1}{1-\sigma} + \mu \frac{G_t^{1-\eta} - 1}{1-\eta} + \beta \mathrm{E}_t \left[V(z_{t+1}, K_{t+1}) \right] \right],$$

subject to[4]

$$K_{t+1} = (1-\delta) K_t + e^{z_t} K_t^\alpha - C(z_t, K_t, G_t) - G_t,$$

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[3] There is a typo in Ambler and Pelgrin (2010, Table 1) in which their depreciation rate is reported to be 0.025.

[4] When optimizing the Bellman equation we substitute the constraint into the continuation value and maximize with respect to G_t.

computing conditional expectations using Gauss-Hermite quadrature. This maximization yields the policy rule for government spending, $G(z_t, K_t)$, and the associated value function, $V(z_t, K_t)$. From $G(z_t, K_t)$ and $C(z_t, K_t, G_t)$ we compute $C(z_t, K_t)$. For GEE, at each node for capital and technology we use direct iteration over equations (14), (17), and (11), again computing conditional expectations using Gauss-Hermite quadrature. For both VFI and GEE, the weights in the Chebyshev polynomials are constructed through Chebyshev-regression with capital and technology constrained to the intervals $K_t \in [5, 15]$ and $z_t \in \left[-\frac{3\sigma_\epsilon^2}{1-\rho^2}, \frac{3\sigma_\epsilon^2}{1-\rho^2} \right]$. To solve the model under its benchmark parameterization we set $n^z = 6$ and $n^k = 9$, and use 50 Chebyshev nodes for the capital stock and 21 Chebyshev nodes for aggregate technology. For the quadrature step, 21 Gauss-Hermite nodes were used.

For the parameterized shadow prices (PSP) method we parameterized the shadow prices according to[5]

$$\Phi(z_t, K_t) \simeq \widetilde{\Phi}(z_t, K_t) = e^{(\lambda_1 + \lambda_2 \ln(K_t) + \lambda_3 z_t)}, \tag{28}$$

$$\Psi(z_t, K_t) \simeq \widetilde{\Psi}(z_t, K_t) = e^{(\theta_1 + \theta_2 \ln(K_t) + \theta_3 z_t)}, \tag{29}$$

and performed Monte Carlo integration using $1,000,000$ simulated observations.

To evaluate the accuracy of each solution we computed the Euler-equation errors (Judd, 1992)

$$EE(z_t, K_t) = 1 - \frac{\left\{ \beta E_t \left[C(z_{t+1}, K_{t+1})^{-\sigma} \left(1 + \left(1 - \frac{G(z_{t+1}, K_{t+1})}{e^{z_{t+1}} K_{t+1}^\alpha - \delta K_{t+1}} \right) \left(\alpha e^{z_{t+1}} K_{t+1}^{\alpha-1} - \delta \right) \right) \right] \right\}^{-\frac{1}{\sigma}}}{C(z_t, K_t)}, \tag{30}$$

employing uniform grids for capital and technology over their intervals and using 1000 grid-points in each dimension.

4.1 Results comparison

Table 2 presents some summary statistics from the solutions; the stochastic steady state values were computing by simulating data ($1,000,000$ observations) from each solution and taking the unconditional mean.

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[5]Equations (28) and (29) describe Ambler and Pelgrin's first-order approximation; they also consider a second-order approximation, obtaining very similar results.

Table 2: Stochastic steady state solutions and accuracy

Variable	VFI	GEE	PSP
Output	1.92262	1.92262	1.92262
Consumption	1.15960	1.15960	1.15959
Government spend.	0.32907	0.32906	0.32910
Investment	0.43396	0.43396	0.43394
Capital	8.67923	8.67921	8.67889
$log_{10} \|EE\|_{\infty}$	-6.75093	-7.41723	-2.06101

For the PSP method, the approximated shadow prices that we obtained were

$$\widetilde{\Phi}\left(z_{t}, K_{t}\right) = e^{(0.923-0.495\ln(K_{t})-0.473z_{t})},$$

$$\widetilde{\Psi}\left(z_{t}, K_{t}\right) = e^{(1.085-0.544\ln(K_{t})-0.457z_{t})},$$

which are very similar to the first-order solution presented in Ambler and Pelgrin (2010, Table 2).[6]

Comparing the solutions shown in Table 2, it is clear that the three procedures produce very similar results, at least in terms of the model's stochastic steady state. To the extent that there are differences, however, it is the PSP method whose results differ, which is consistent with the PSP method being generally less accurate and having larger Euler-equation errors.[7] It should be emphasized, however, that the differences among the three solutions are minor, a finding that carries over to the unconditional densities shown in Figure 1. Indeed, looking at Figure 1, although PSP produces Euler-equation errors (panel C) that are somewhat larger than either VFI or GEE, because these errors are not particularly systematic (such as being mostly of the same sign) they largely wash-out when computing the unconditional densities (panels D—I). To the extent that differences between PSP and the other two methods are apparent, they reside chiefly in the decision rule for government spending (panel B), where it is apparent that PSP tends to understate government spending when capital is large.[8]

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[6] The results that we obtain from a second-order version in which equations (28) and (29) also contain the squares of $ln\left(K_{t}\right)$ and z_{t} and the interaction between $ln\left(K_{t}\right)$ and z_{t}, are very similar to those presented in Table 2 and Figure 1.

[7] The Euler-equation errors presented in Figure 1 differ from those presented in Ambler and Pelgrin (2010). We suspect that the source of this difference resides in the procedure used to numerically integrate the technology shock. If we simply set the technology shock to its unconditional mean when computing the Euler-equation errors, then we obtain results very similar to theirs.

[8] The decision rules for shown in Figure 1 (panels A and B), depict consumption and government spending, respectively, as a function of capital, holding $z_{t} = 0$.
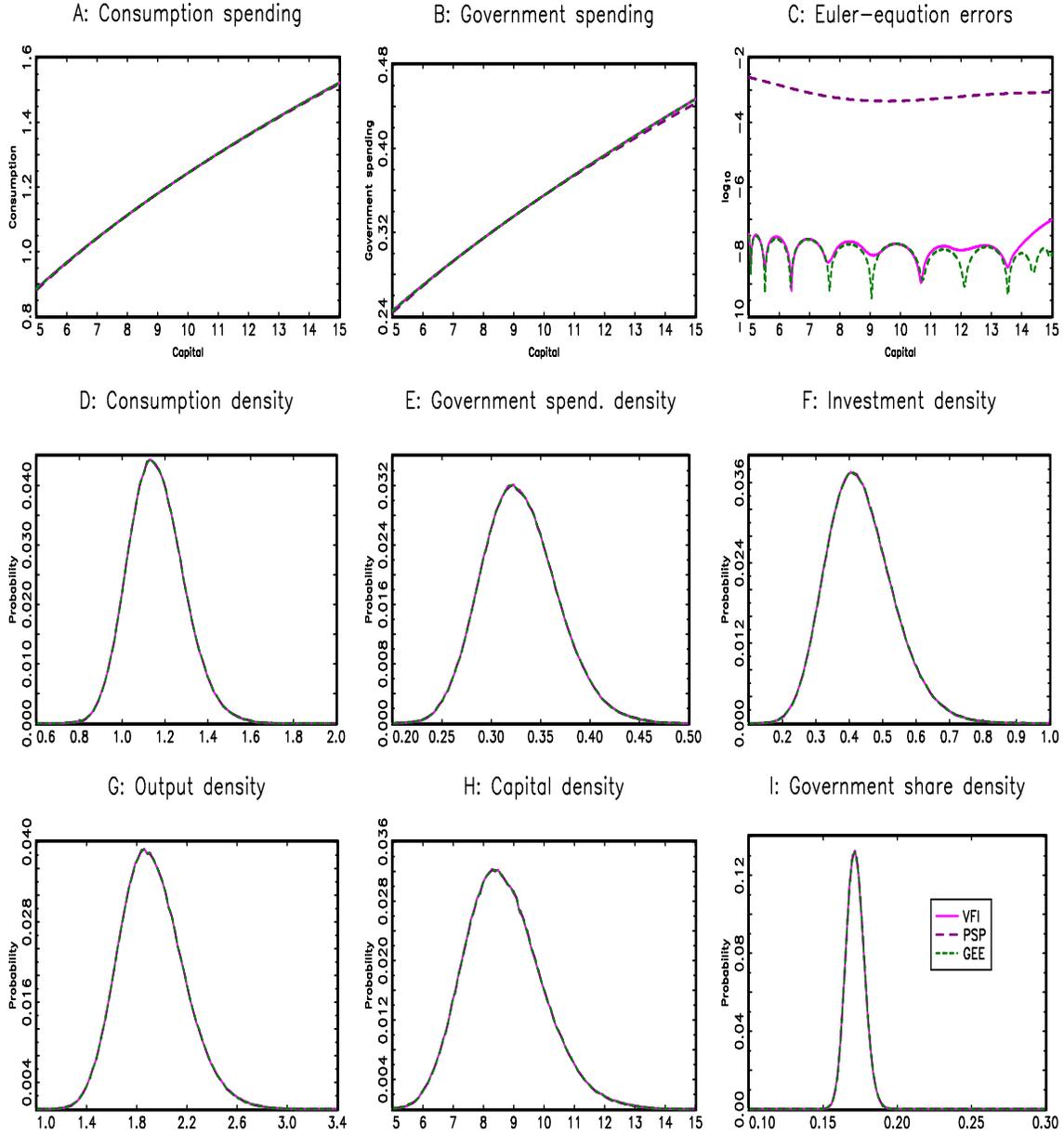
A: Consumption spending   B: Government spending   C: Euler−equation errors

D: Consumption density   E: Government spend. density   F: Investment density

G: Output density   H: Capital density   I: Government share density

Figure 1: Comparing solutions

4.2   The numerical accuracy of the PSP approach

We saw in Figure 1 that the PSP solution method obtained using parameterized expectations was less accurate in terms of Euler equation errors that either GEE or VFI. To explore the reason for this decline in accuracy, we compute an alternative PSP solution (PSP-alt) in which the

approximating function is a Chebyshev polynomial and expectations are computed using Gauss-Hermite quadrature rather than Monte Carlo integration. For PSP-alt, therefore, we replace equations (28) and (29) with polynomials in the form of equation (27), and use the same grid as used for VFI and GEE. The results are shown in Figure 2.
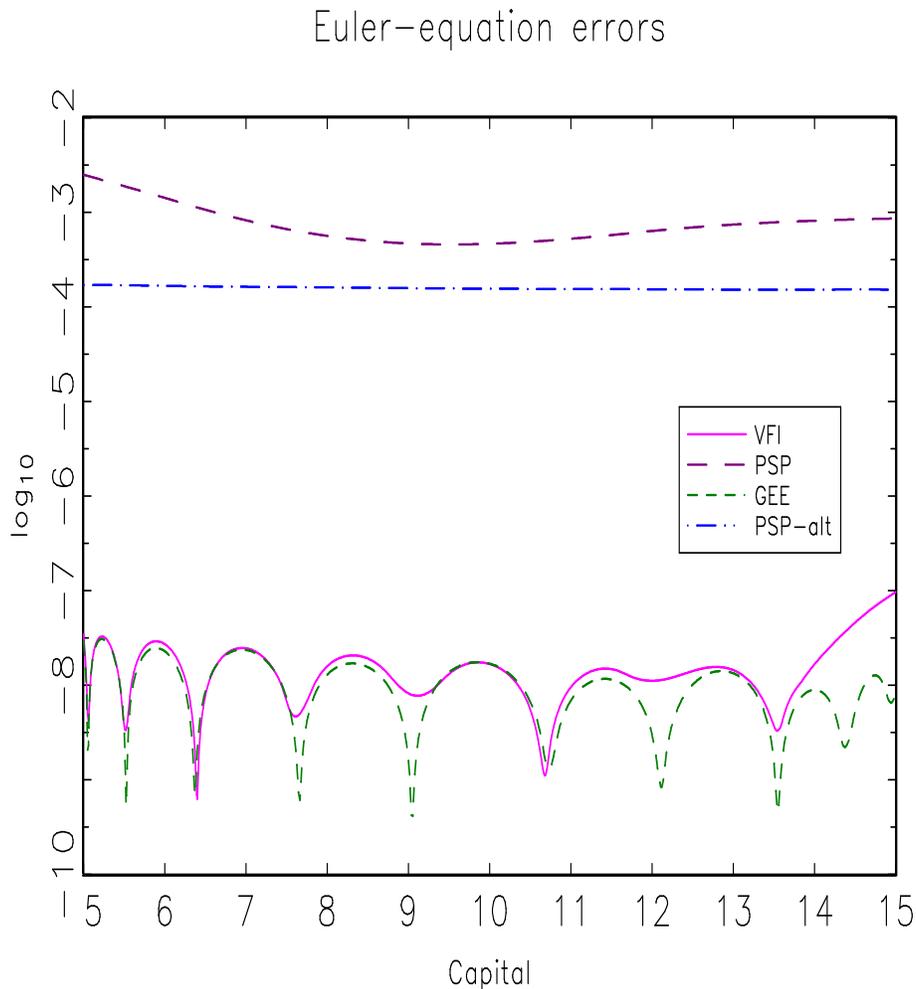

Figure 2: Numerical accuracy of PSP solution method

It is clear from Figure 2 that PSP-alt remains inferior in terms of Euler-equation accuracy than either GEE or VFI, but improves upon the benchmark PSP-solution described in Section 3.3 (which follows Ambler and Pelgrin, 2010). This implies, therefore, that the decline in numerical accuracy associated with PSP stems less from the choice of approximating function or from the

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method of integration, and more on the object being approximated. Specifically, comparing PSP-alt with GEE the essential difference is that GEE approximates the decision rules for consumption and government spending whereas PSP-alt approximates convolutions of these decision rules as part of a more general function.

5 More sophisticated policy-problems

In this section we consider two policy problems that are more sophisticated than the benchmark model considered above. The first of these policy problems introduces a constraint on government spending (as a share of output), a constraint that is occasionally binding in that it binds in some regions of the state-space, but not in others. In many respects, this occasionally binding constraint is not unlike those that appear in monetary policy models with a zero-lower bound on the nominal interest rate or in models with occasionally binding collateral constraints. The second of these policy problems assumes that the fiscal authority is a risk-sensitive decisionmaker. We consider these two variations on the benchmark model because they each pose problems for GEE and PSP—particularly risk-sensitivity—but can be accommodated easily through VFI. In particular, although models containing inequality constraints like that described in section 5.1 can be solved using the PSP and GEE solution methods, using the techniques described in Christiano and Fisher (2000), for example, these techniques are somewhat more diffcult to implement than simply imposing box-constraints within a hill-climber, which is all that is required for VFI. In regard to models with risk-sensitive preferences, first-order methods such as GEE and PSP are generally inappropriate for such problems precisely because they do not retain the level of the value function.

5.1 Constraints on government spending or taxes

Using VFI it is straightforward to impose inequality constraints on government spending by employing a constrained hill-climber when optimizing the fiscal authority's value function with respect to G_t. Accordingly, if one wants to constrain government spending as a share of output, such as

$$\kappa_h e^{z_t} K_t^\alpha \geq G_t \geq \kappa_l e^{z_t} K_t^\alpha, \tag{31}$$

where $\kappa_h > \kappa_l > 0$, then one simply determines the upper and lower bound on G_t for each node in the state space and determines $G(z_t, K_t)$ and $V(z_t, K_t)$ using constrained optimization.

Inequality constraints on the tax rate can be similarly accommodated. For example, to constrain τ_t to reside in the interval $\tau_t \in [\tau_l, \tau_h]$, we translate this inequality constraint on the tax rate into one on government spending, as per

$$\tau_h \left(e^{z_t} K_t^\alpha - \delta K_t \right) \geq G_t \geq \tau_l \left(e^{z_t} K_t^\alpha - \delta K_t \right), \tag{32}$$

and then, as before, maximize the value function at each node in the state space using a constrained hill-climber.

For illustrative purposes, the model that we consider is one in which government spending is constrained to be no less than 16.5 percent of output, with this percent chosen so that the spending constraint would bind occasionally, but not predominantly. We refer to this model as the fiscally constrained model.

5.2 Risk-sensitive preferences

VFI can also be used to solve decision problems in which the policymaker has risk-sensitive preferences, perhaps motivated by ambiguity aversion or by an aversion to model uncertainty (Hansen and Sargent, 2008). In place of equation (12), a risk-sensitive fiscal authority will conduct fiscal policy based on the Bellman equation

$$V \left(z_t, K_t \right) = \max_{G_t, K_{t+1}} \left[\frac{C \left(z_t, K_t, G_t \right)^{1-\sigma} - 1}{1 - \sigma} + \mu \frac{G_t^{1-\eta} - 1}{1 - \eta} + \frac{\beta}{\theta} \ln \left[\mathrm{E}_t e^{[\theta V(z_{t+1}, K_{t+1})]} \right] \right], \tag{33}$$

where $\theta < 0$ is the risk-sensitivity parameter and the constraint continues to be given by equation (10). An application of L'Hôpitals' rule establishes that equation (12) is restored in the limit as $\theta \uparrow 0$ while the effects of $\theta < 0$ are to distort the continuation value in the Bellman equation, a distortion arising from the fiscal authority's aversion to risky life-time utility. Because the value function is an object that VFI retains and delivers, risk-sensitive preferences are straightforward to analyze. To illustrate this point, and to examine the effects that risk-sensitive preferences have on Markov-perfect fiscal policy, we consider a decision problem for which the fiscal authority's Bellman equation is described by equation (33) with $\theta = -1.0$. We refer to this model as the risk-sensitive preferences model.

5.3 First-best

As a baseline against which to contrast the Markov-perfect equilibria obtained from the benchmark model, the fiscally constrained model, and the risk-sensitive preferences model, we also solve

for the first-best equilibrium in the benchmark model. To compute this first-best equilibrium we formulate the decision problem for the fiscal authority in terms of the Bellman equation

$$V\left(z_{t}, K_{t}\right) = \max_{\{C_{t}, G_{t}, K_{t+1}\}} \left[\frac{C^{1-\sigma} - 1}{1 - \sigma} + \mu \frac{G_{t}^{1-\eta} - 1}{1 - \eta} + \beta \mathrm{E}_{t}\left[V\left(z_{t+1}, K_{t+1}\right)\right] \right], \tag{34}$$

with the constraint given by equation (10) and with technology evolving according to equation (4). Although there are two choice variables, this is a standard dynamic programming problem and can be solved using the VFI strategy described above. For the benchmark model, the first-best equilibrium must satisfy the resource constraint

$$K_{t+1} = (1 - \delta) K_{t} + e^{z_{t}} K_{t}^{\alpha} - C_{t} - G_{t},$$

and the first-order conditions

$$C_{t}^{-\sigma} = \beta \mathrm{E}_{t}\left[C_{t+1}^{-\sigma}\left(1 - \delta + \alpha e^{z_{t+1}} K_{t+1}^{\alpha-1}\right)\right], \tag{35}$$

$$C_{t}^{-\sigma} = \mu G_{t}^{-\eta}. \tag{36}$$

We use equation (35) to construct the Euler-equation errors in our accuracy test, which, because it is not imposed in obtaining the solution, leads to a relatively stringent test of accuracy.

5.4   Results

In this section we solve for the Markov-perfect equilibrium of the benchmark model, the fiscally constrained model, and the risk-sensitive preferences model. We contrast these equilibria with the first-best equilibrium obtained from the benchmark model. Table 3 and Figure 2 contain a summary of the main results. The "deterministic" steady state results shown in Table 3 correspond to the steady state in a Markov-perfect equilibrium of a deterministic version of the benchmark model. However, because risk-sensitivity generates no risk-adjustment when the model is deterministic and because the government spending constraint that we impose in the fiscally constrained model does not bind at the deterministic steady state, the deterministic steady state values reported apply equally to the fiscally constrained model and to the risk-sensitive preferences model.

Table 3: Steady state solutions and numerical accuracy					
Variable	Deterministic	Benchmark	Fiscally const.	Risk-sensitive	First-best
Output	1.90236	1.92262	1.9223076	1.92869	1.96881
Consumption	1.15008	1.15960	1.1587733	1.19650	1.15329
Government spend.	0.32575	0.32907	0.3298144	0.29366	0.34599
Investment	0.42653	0.43396	0.4337198	0.43853	0.46953
Capital	8.53053	8.67923	8.6744408	8.77060	9.39064
$log_{10} \|EE\|_\infty$	-7.46797	-6.75093	-4.97567	-7.03006	-4.75445

Table 3 shows that the deterministic steady state for capital is just over 8.53. Allowing for stochastic aggregate technology, as per the benchmark model, the (stochastic) steady state for capital rises to about 8.68. This rise in steady-state capital occurs through a standard precautionary saving motive whereby risk-averse households increase their saving and accumulate capital as a buffer to self-insure against adverse technology shocks. Consistent with a higher steady state capital stock, the steady state values for output, consumption, and investment are all higher in the benchmark model than they are in its deterministic counterpart.

The constraint that government spending as a share of output be no smaller than 0.165 has little effect on the model's stochastic steady state, which is very similar to that of the benchmark model, but it does lead to a decline in numerical accuracy. Nonetheless, with the log_{10} of the maximum Euler-equation error at about -5 the maximal solution error amounts to about one dollar out of every one hundred thousand dollars spent, which certainly seems acceptable. Comparing the benchmark model to the risk-sensitive preferences model, it is clear that the fiscal authority's risk-sensitivity is having a profound effect on fiscal policy. Specifically, at the (stochastic) steady state the risk-sensitive fiscal authority chooses less government spending than does the fiscal authority in the benchmark model. With less government spending the risk-sensitive fiscal authority also imposes a smaller income tax rate, reducing some of the disincentive households's face to accumulate capital. As a consequence, the (stochastic) steady state for capital in the risk-sensitive preferences model is almost 8.8, somewhat higher than in the benchmark model. Table 3 also presents summary statistics for the first-best equilibrium, quantifying the effects of distortionary taxation and the time-inconsistency problem that these distortions produce.
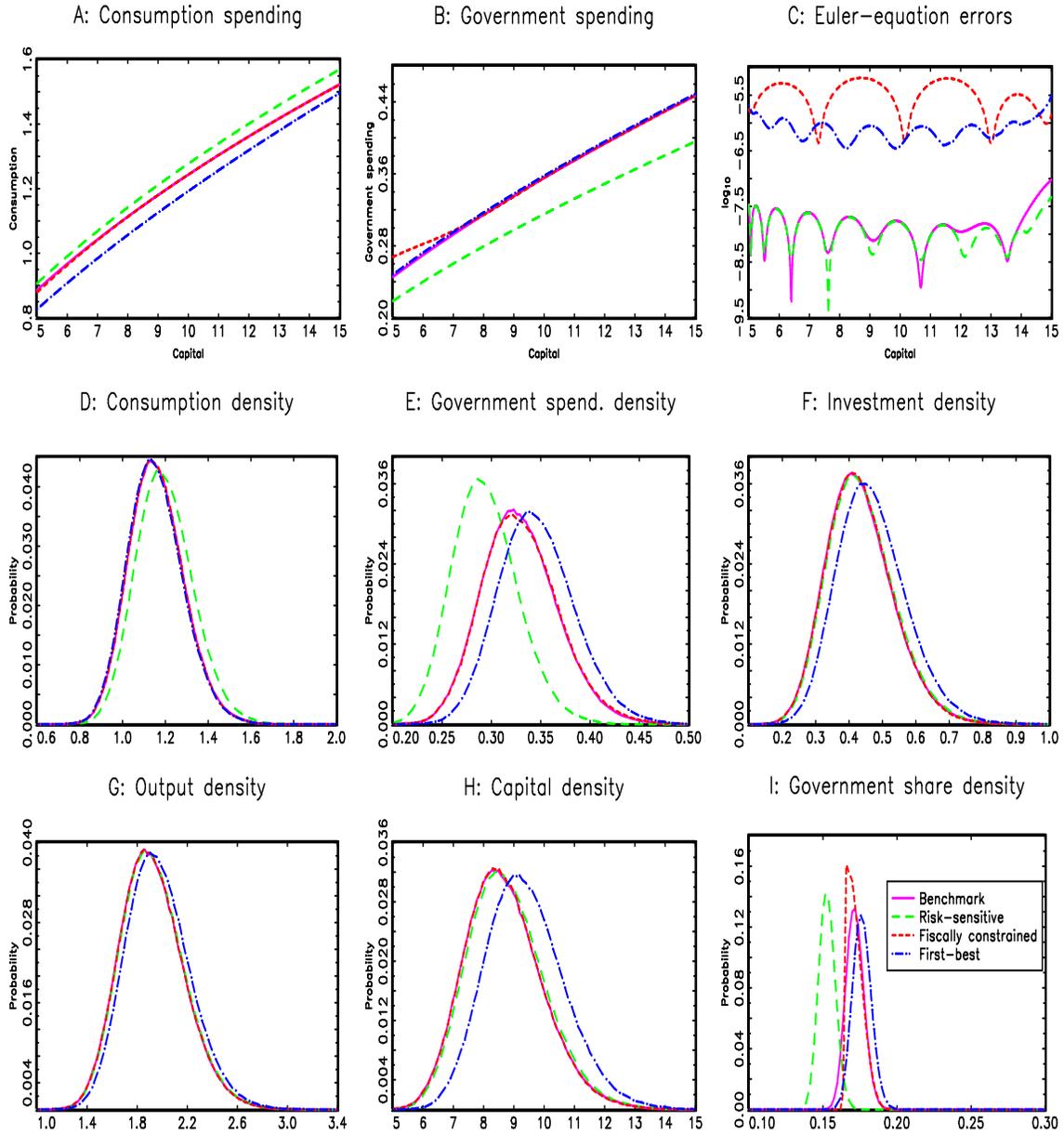
Figure 3: Solution and accuracy

Where Table 3 presents information about steady state outcomes, Figure 3 presents the consumption function, the government spending function, the densities of the key model-variables, and additional information about numerical accuracy. As earlier, we construct the densities by simulating data from each solution, we report the decision rules for consumption and government spending holding $z_t = 0$, and we use equation (30) to compute the Euler-equation errors (suitably

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modified in the case of the first-best equilibrium).

Perhaps unsurprisingly, the consumption functions for the benchmark model and the fiscally constrained model are very similar. The differences between these two models are reflected more prominently in their respective government spending functions, where the constraint on government spending binds occasionally to force higher government spending in the fiscally constrained economy. The effects of risk-sensitivity are evident in both the consumption function and the government spending function. Because the risk-sensitive fiscal authority is especially concerned about risk, she wishes households to build up a buffer-stock of capital, keeping government spending and hence income taxes low in order to encourage households to do so. Of course, the income effect associated with the lower lifetime tax liability (the model is Ricardian) induces households to also consume more, leading households to consume more in the risk-sensitive model relative to the benchmark model.

Interestingly, each of the models produces similar densities for output and, to a lesser extent, for consumption. The four models differ importantly, however, with respect to the densities for investment, capital, government spending, and government spending as a share of output. With distortionary taxes, government spending is too low, both in absolute terms and as a share of output, relative to the first-best equilibrium. The constraint on government spending in the fiscally constrained economy is evident in the density for government spending as a share of output, but has relatively little effect on the density for government spending itself.

Having shown the densities for output, consumption, etc, in Figure 3, Figure 4 presents the impulse response functions for the key model variables following a positive one standard deviation technology shock; Figure 5 displays analogous responses, but for a negative one standard deviation technology shock. Looking at Figure 4, there are notable differences in how the various model's respond, particularly in regard to government spending and consumption. In the fiscally constrained economy, with output rising in response to the technology shock the constraint on government spending binds pushing up government spending at the expense (primarily) of lower private consumption. At the same time, interestingly, the benchmark equilibrium, the risk-sensitive equilibrium, and the first-best equilibrium respond similarly to the shock.
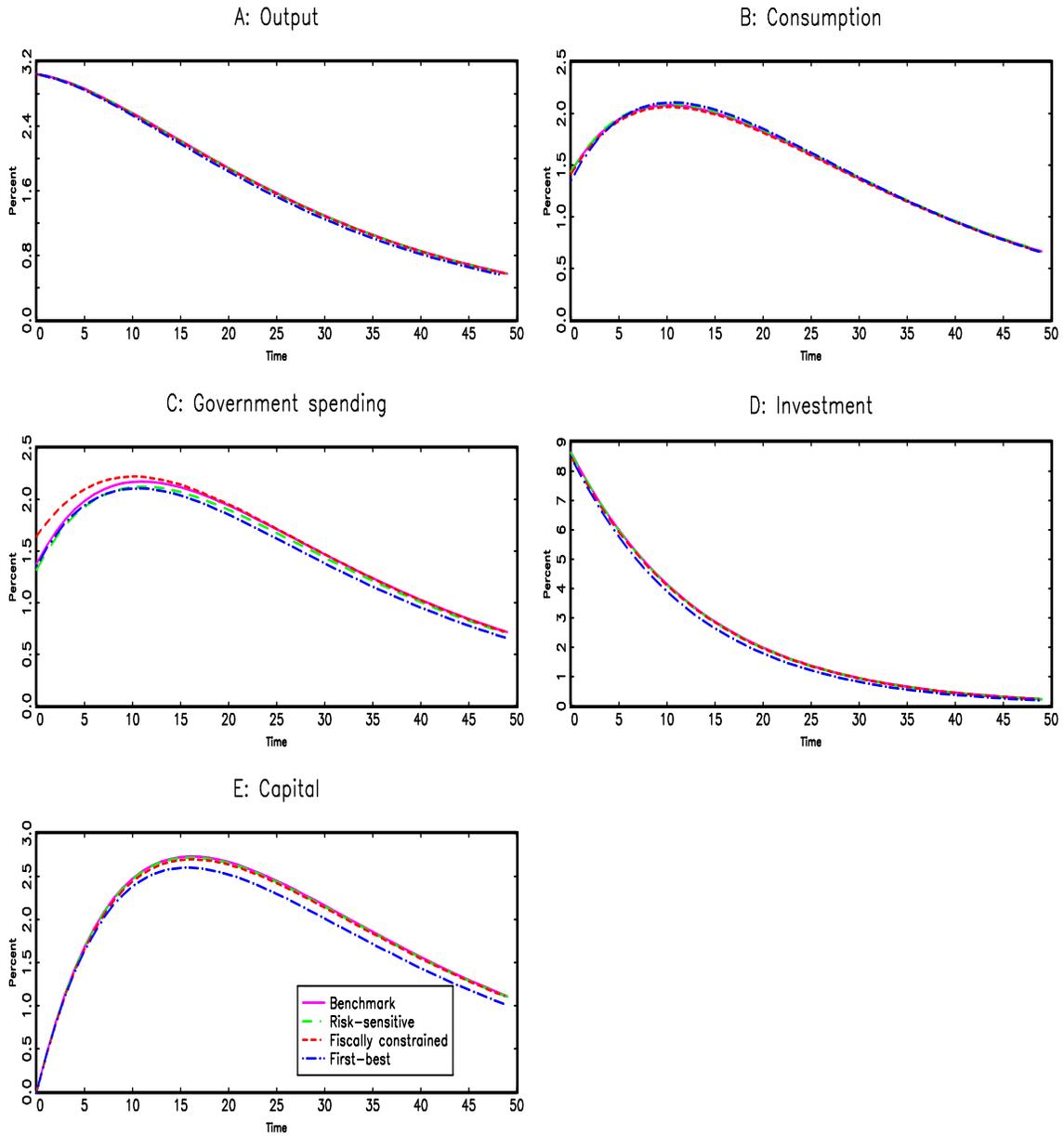
Figure 4: Responses to a positive 1 s.d. technology shock

In response to a negative technology shock, Figure 5 reveals some asymmetries relative to the responses to a positive shock shown in Figure 4. In particular, unlike a positive technology shock, the constraint on government spending does not bind following a negative technology shock and there are greater differences between the first-best equilibrium and the benchmark model. Other than for the fiscally constrained economy, for which there is a clear asymmetry in the government

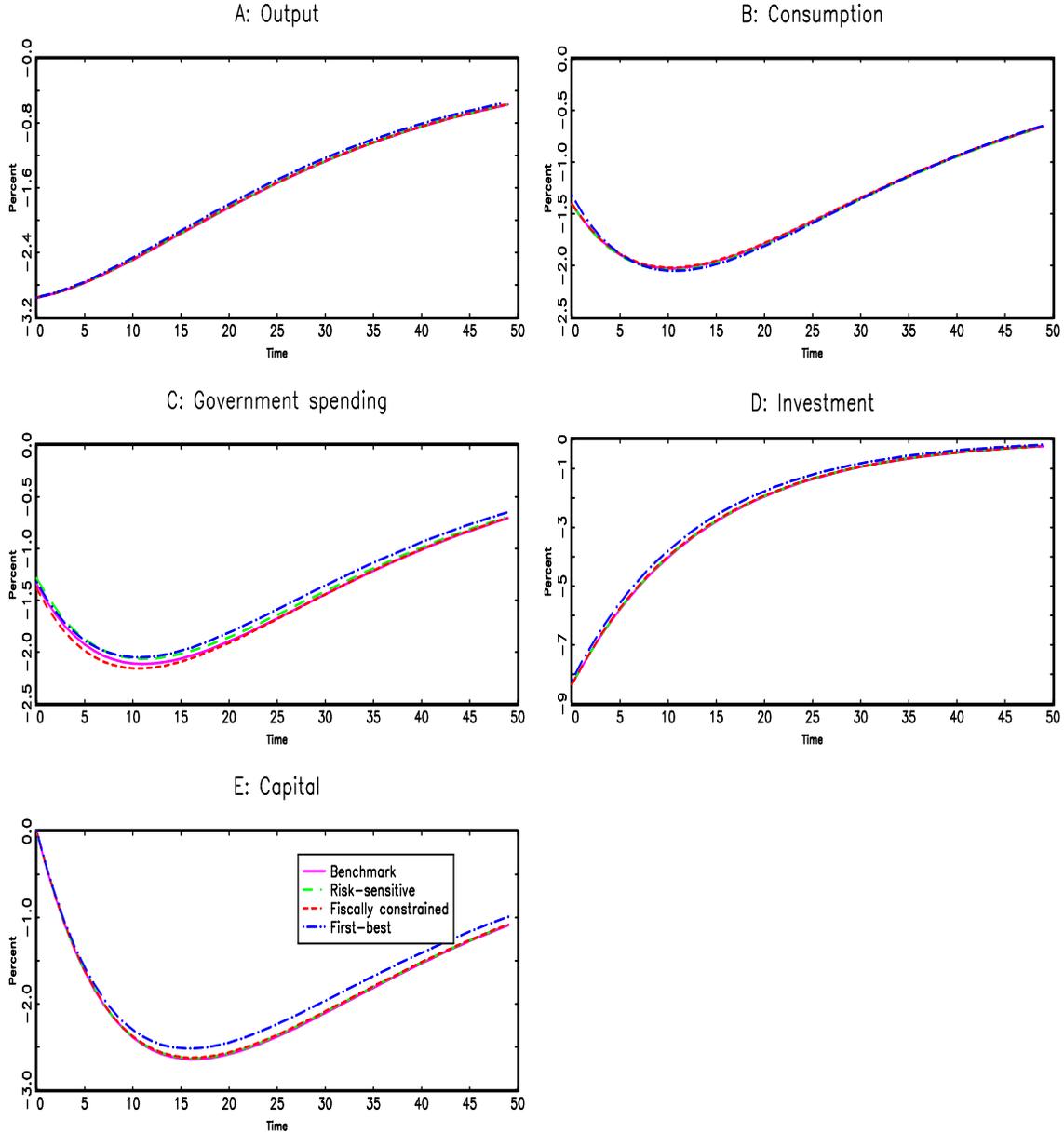spending response, the models' responses are relatively symmetric

Figure 5: Responses to a negative 1 s.d. technology shock

To better identify asymmetries in the impulse response functions we plot in Figure 6 the sum of the responses to the positive and negative shock for each variable. To the extent that each model is asymmetric the sum of the responses differs from zero. Looking at Figure 6 (panel E) we see a notable asymmetry in the capital stock's reponses to technology shocks, with this asymmetry

propagated through capital role as a state variable into the remaining variables. In addition to this asymmetry, Figure 5 also reveals the asymmetries present in the fiscally constrained economy and shows how the occasionally binding constraint on government spending induces asymmetric behavior in consumption and investment.
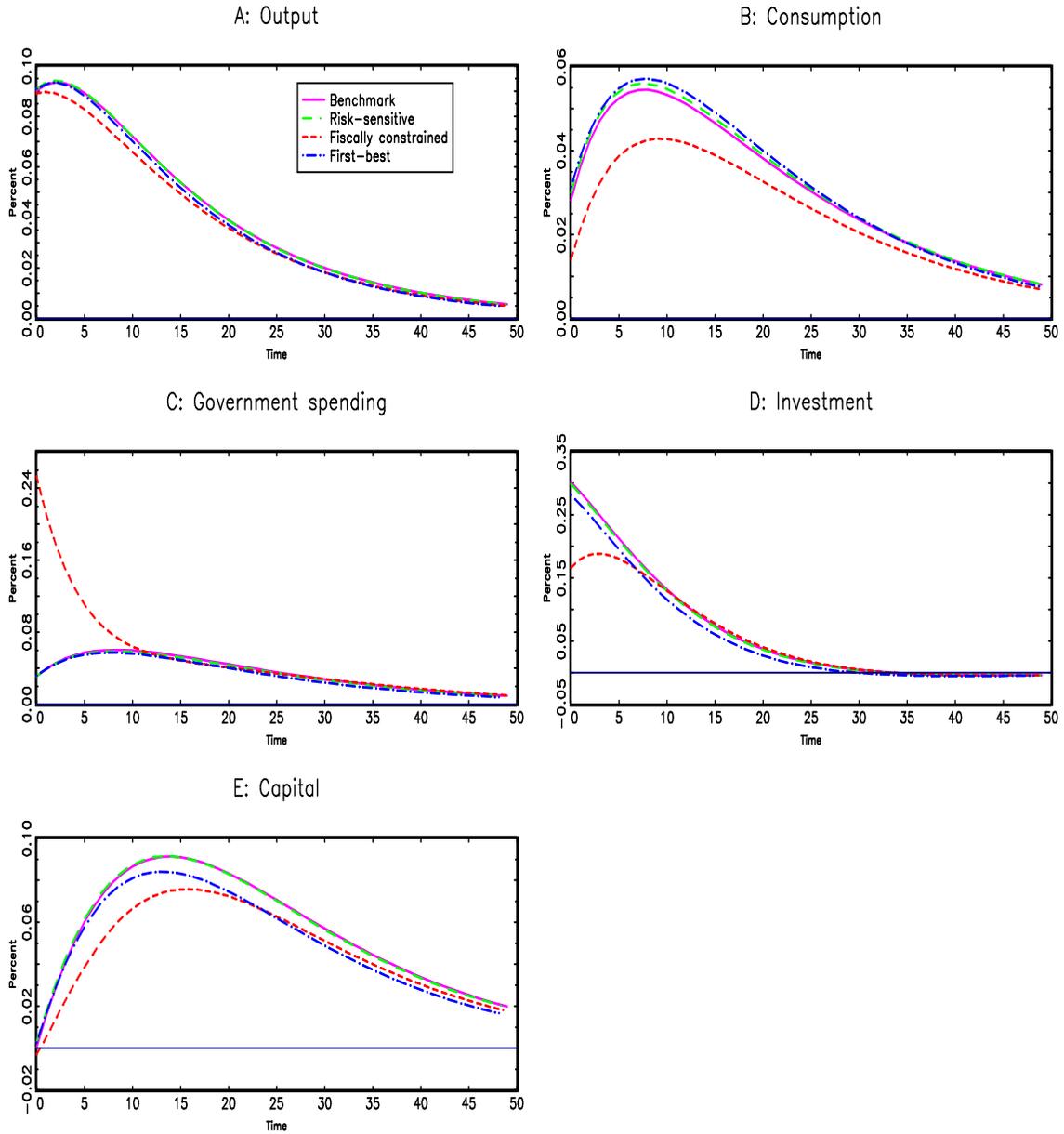

Figure 6: Impulse response asymmetries

6   Discussion

The models solved above are relatively small in that they contain only one endogenous state variable and one shock. With only two state variables and the need to integrate over a single shock these models can be solved relatively quickly by all three methods, although with varying accuracy. Excluding coding time, which is an important consideration, GEE and PSP-alt were the quickest, followed by VFI and then PSP.[9] However, each of these methods suffers from the curse of dimensionality. As a consequence, solving larger models, ones containing a handful of state variables or more, can be very time-consuming. For such models, it may be advantageous to employ complete polynomials rather than tensor-product polynomials or to use sparse-grid methods (Smolyak, 1963), as described in Malin, Krueger, and Kubler (2011), for example, or monomial-methods (Pichler, 2011).

7   Conclusion

This paper has presented and compared three methods for computing Markov-perfect optimal policies in nonlinear business cycle models. Of these methods, two were based on Euler equations while the third was based on value function iteration. We illustrated these solution methods by applying them to a canonical business cycle model of government-good provision in which a fiscal authority must determine optimally the provision of government goods, subject to distortionary taxation, a balanced-budget constraint, and while lacking a commitment technology. For this benchmark model we found that all three methods worked well, that the use of generalized Euler equations and value function iteration gave essentially identical results, and that the parameterized shadow price method was the less accurate of the three. We further showed that the value function iteration solution method could easily be extended to accommodate model features such as risk-sensitive preferences and certain forms of occasionally binding constraints, such as bounds on government spending or the tax rate.

Although our application focused on fiscal policy, many economic decision problems can be tackled using the solution procedures described in this paper. In particular, problems relating to monetary policy design, both with and without a zero-bound on nominal interest rates, monetary

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[9]This ordering simply reflects our experience with these methods, without attempting to optimize them for speed. A comprehensive study of solution times, one that optimized the solution methods, conditioned them upon identical initial conditions (to the extent possible), and the requirement that they produce the same level of accuracy, is beyond the scope of this paper.

and fiscal policy coordination, exchange rate management, and international lending. These solution procedures can also be used to examine the effects of risk-sensitivity on allocations and asset prices in economies where policymakers cannot commit.

References

[1] Adam, K., and R. Billi, (2007), "Discretionary Monetary Policy and the Zero-Bound on Nominal Interest Rates," *Journal of Monetary Economics*, 54, pp. 728–754.

[2] Ambler, S., and F. Pelgrin, (2010), "Time-Consistent Control in Nonlinear Models," *Journal of Economic Dynamics and Control*, 34, pp. 2215–2228.

[3] Anderson, G., Kim, J., and T. Yun, (2010), "Using a Projection Method to Analyze Inflation Bias in a Micro-Founded Model," *Journal of Economic Dynamics and Control*, 34, pp. 1572–1581.

[4] Backus, D., and J. Driffill, (1986), "The Consistency of Optimal Policy in Stochastic Rational Expectations Models," Centre for Economic Policy Research Discussion Paper #124.

[5] Benhabib, J., and A. Rustichini, (1997), "Optimal Taxes Without Commitment," *Journal of Economic Theory*, 77, pp. 231–259.

[6] Benveniste, L., and J. Scheinkman, (1979), "On the Differentiability of the Value Function in Dynamic Models of Economies," *Econometrica*, 47, 3, pp. 727–732.

[7] Christiano, L., and J. Fisher, (2000), "Algorithms for Solving Dynamic Models with Occasionally Binding Constraints," *Journal of Economic Dynamics and Control*, 24, pp. 1179–1232.

[8] Clarida, R., Galí, J., Gertler, M., (1999), "The Science of Monetary Policy: A New Keynesian Perspective," *Journal of Economic Literature*, 37, pp. 1661–1707.

[9] Cohen, D., and P. Michel, (1988), "How Should Control Theory Be Used to Calculate a Time-Consistent Government Policy," *Review of Economic Studies*, LV, pp. 263–274.

[10] Currie, D., and P. Levine, (1985), "Macroeconomic Policy Design in an Interdependent World," in Buiter, W., and R. Marston, (eds), *International Economic Policy Coordination*, Cambridge University Press, Cambridge.

[11] Currie, D., and P. Levine, (1993), *Rules, Reputation and Macroeconomic Policy Coordination*, Cambridge University Press.

[12] Dennis, R., (2007), "Optimal Policy in Rational Expectations Models: New Solution Algorithms," *Macroeconomic Dynamics*, 11, pp. 31-55.

[13] Domínguez, B., (2007), "Public Debt and Optimal Taxes Without Commitment," *Journal of Economic Theory*, 135, pp. 159–170.

[14] Hansen, L., and T. Sargent, (2008), *Robustness*, Princeton University Press, Princeton.

[15] Inada, K-I., (1963), "On a Two-Sector Model of Economic Growth: Comments and a Generalization," *Review of Economic Studies*, 30, pp. 119-127.

[16] Judd, K., (1992), "Projection Methods for Solving Aggregate Growth Models," *Journal of Economic Theory*, 58, pp. 410–452.

[17] Klein, P., Krusell, P., and J-V. Rios-Rull, (2008), "Time-Consistent Public Policy," *Review of Economic Studies*, 75, pp. 789–808.

[18] Klein, P., and J-V. Rios-Rull, (2003), "Time-Consistent Optimal Fiscal Policy," *International Economic Review*, 44, 4, pp. 1217–1245.

[19] Kydland, F., and E. Prescott, (1977), "Rules Rather than Discretion: The Inconsistency of Optimal Plans," *Journal of Political Economy*, 85, 3, pp. 473–491.

[20] Malin, B., Krueger, D., and F. Kubler, (2011), "Solving the Multi-Country Real Business Cycle Model using a Smolyak-Collocation Method," *Journal of Economic Dynamics and Control*, 35, pp. 229–239.

[21] Marcet, A., and G. Lorenzoni, (1999), "The Parameterized Expectations Approach: Some Practical Issues," in Marimon, R., and A. Scott, (eds), *Computational Methods for the Study of Dynamic Economies*, Oxford University Press, Oxford.

[22] Martin, F., (forthcoming), "Policy and Welafre Effects of Within-Period Commitment," *Macroeconomic Dynamics*, forthcoming.

[23] Nakata, T., (2012), "Optimal Fiscal and Monetary Policy with Occasionally Binding Zero-Bound Constraints," Manuscript (version dated January 12, 2012).

[24] Niemann, S., Pichler, P, and G. Sorger, (2008), "Optimal Fiscal and Monetary Policy without Commitment," Manuscript (version dated June 16, 2008).

[25] Niemann, S., Pichler, P, and G. Sorger, (2009), "Inflation Dynamics under Optimal Discretionary Fiscal and Monetary Policies," Manuscript (version dated December 17, 2009).

[26] Ortigueira, S., (2006), "Markov-Perfect Optimal Taxation," *Review of Economic Dynamics*, 9, pp. 153–178.

[27] Ortigueira, S., and J. Pereira, (2009), "Markov-Perfect Optimal Fiscal Policy: The Case of Unbalanced Budgets," Manuscript (version dated November 4, 2009).

[28] Oudiz, G., and J. Sachs, (1985), International Policy Coordination in Dynamic Macroeconomic Models," in Buiter, W., and R. Marston, (eds), *International Economic Policy Coordination*, Cambridge University Press, Cambridge, 275-319.

[29] Pichler, P., (2011), "Solving the Multi-Country Real Business Cycle Model using a Monomial Rule Galerkin Method," *Journal of Economic Dynamics and Control*, 35, pp. 240–251.

[30] Smolyak, S., (1963), "Quadrature and Interpolation Formulas for Tensor Products of Certain Classes of Functions," *Soviet Mathematics*, Doklady, 4, pp. 240–243.

[31] Söderlind, P., (1999), "Solution and Estimation of RE Macromodels with Optimal Policy," *European Economic Review*, 43, 813-823.

[32] Stockman, D., (2001), "Balanced-Budget Rules: Welfare Loss and Optimal Policies," *Review of Economic Dynamics*, 4, pp. 438–459.

[33] Woodford, M., (2003), *Interest and Prices: Foundations of a Theory of Monetary Policy*, Princeton University Press, Princeton, NJ.