Advanced Higher Physics:Assignment Support

Astronomy & Physics Education Group

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Determination of g by Rolling Sphere

Introduction

The simple harmonic motion (SHM) of a spherical ball on a spherical curved mirror surface can be used to determine a value for g. In Figure 1 below, we assume that the spherical ball of radius r can roll, without slipping, on the surface of the spherical mirror of radius R.

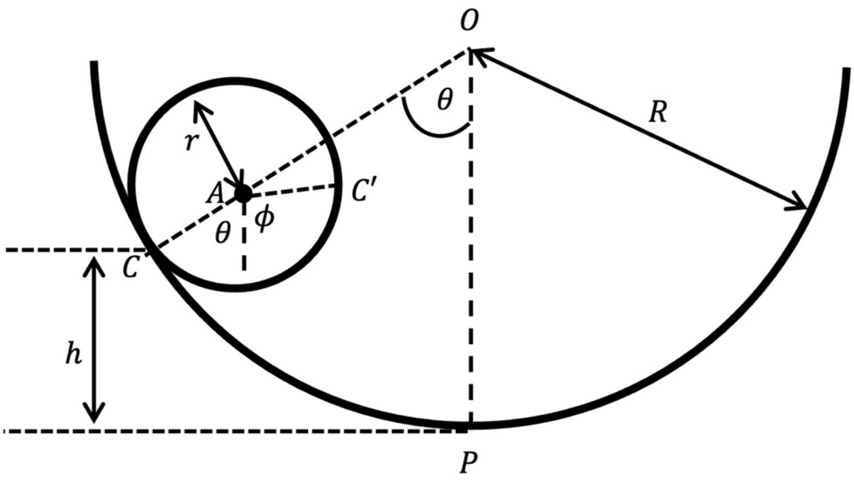

Figure 1: Schematic of rolling sphere experiment (not to scale)

If we assume that the ball rolls without slipping, then the distance along the arc CC' is the same as the distance along arc CP.

If θ is the angle subtended at the centre of the circle moving from C to P then $CP = R\theta$, provided we are measuring the angle in radians. Similarly, we can say that $CC' = (\theta + \phi)r$. And since both arcs are equal ...

$$r(\theta + \phi) = R\theta$$
$$\Rightarrow r\phi = (R - r)\theta$$

[1]

[1] is called the "rolling condition".

If we assume that the surface of the mirror can be considered to be frictionless, then a ball rolling back and forth across that surface will undergo simple harmonic motion. To determine the mathematical form of that motion, we should consider energy conservation:

$$\text{Kinetic energy} + \text{potential energy} = \text{Constant}$$

(Again, assuming no friction.)

Kinetic energy can be split into two parts:

1. *translational* energy of the sphere given by $K_t = \frac{1}{2}mv^2$, where v is the velocity of the centre of the sphere, A, through space, and
2. *rotational* kinetic energy of the sphere about its centre given by $K_r = \frac{1}{2}I\omega^2$, where I is the moment of inertia of the sphere about its centre and ω is the angular velocity of the sphere about its centre.

The linear velocity of the sphere, v, as it travels across the mirror is equivalent to a point like object with the same mass as the sphere, moving on a circular trajectory at a distance $R - r$ from the centre of curvature of the mirror, point O. i.e. $v = (R - r)\frac{d\theta}{dt}$

The moment of inertia I of the sphere, assuming its mass is m, about its centre is given by $I = \frac{2}{5}mr^2$.

The sphere itself is rotating about an axis perpendicular to Figure 1 (out of page). Its angular velocity ω, then is the rate of change of ϕ with respect to time, i.e. $\omega = \frac{d\phi}{dt}$.

In moving from C to P the ball moves through a vertical distance h. By considering the angles in Figure 1 we can see that $h = (R - r)(1 - \cos\theta)$. Therefore. the potential energy lost as the ball moves from C to P is $E_P = mg(R - r)(1 - \cos\theta)$.

We can now re-write [1] as:

$$\frac{1}{2}mv^2 + \frac{1}{2}I\left(\frac{d\phi}{dt}\right)^2 + mg(R - r)(1 - \cos\theta) = \text{constant}$$

[2]

Substituting in $v = (R - r)\frac{d\theta}{dt}$ and $I = \frac{2}{5}mr^2$, this becomes ...

$$\frac{1}{2}m(R - r)^2\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{2}\frac{2}{5}mr^2\left(\frac{d\phi}{dt}\right)^2 + mg(R - r)(1 - \cos\theta) = \text{constant}$$

$$\Rightarrow (R - r)^2\left(\frac{d\theta}{dt}\right)^2 + \frac{2}{5}r^2\left(\frac{d\phi}{dt}\right)^2 + 2g(R - r)(1 - \cos\theta) = \text{constant}$$

Now recalling equation [1]: $r\phi = (R - r)\theta \Rightarrow r\frac{d\phi}{dt} = (R - r)\frac{d\theta}{dt}$ ...

$$(R - r)^2\left(\frac{d\theta}{dt}\right)^2 + \frac{2}{5}(R - r)^2\left(\frac{d\theta}{dt}\right)^2 + 2g(R - r)(1 - \cos\theta) = \text{constant}$$

$$\Rightarrow \frac{7}{5}(R - r)^2\left(\frac{d\theta}{dt}\right)^2 + 2g(R - r)(1 - \cos\theta) = \text{constant}$$

$$\Rightarrow \frac{7}{5}(R - r)\left(\frac{d\theta}{dt}\right)^2 + 2g(1 - \cos\theta) = \text{constant}$$

Differentiating this with respect to time gives us ...

$$\Rightarrow \frac{7}{5}(R - r)\frac{d^2\theta^2}{dt^2} + g\sin\theta = \text{constant}$$

and if we assume we have small angles, then this becomes

$$\Rightarrow \frac{7}{5}(R-r)\frac{d^2\theta}{dt^2}^2 + g\theta = \text{constant}$$

$$\Rightarrow \frac{d^2\theta}{dt^2}^2 + \frac{5g}{7(R-r)}\theta = \text{constant}$$

[3]

This describes SHM for variable θ. Comparing with the standard expression for SHM, [3] tells us that the period of oscillations can be given by

$$\boxed{T = 2\pi\sqrt{\frac{7(R-r)}{5g}}}$$

[4]

Recall that for a simple pendulum,

$$T = 2\pi\sqrt{\frac{l}{g}}$$

So we can see that our sphere behaves as if it were a simple pendulum of length $\frac{7(R-r)}{5}$. In practice the radius of curvature of the mirror, R, is much greater than the radii of suitable ball-bearings (i.e. $R \gg r$) and consequently the term $\frac{7(R-r)}{5}$ does not change much for different balls. The square root reduces further this already small variation and so T does not change very significantly with change of ball-bearing radius.

Given the relatively large random uncertainty associated with measuring T using a stopwatch, the variation with radius is likely to be masked by random uncertainty. Consequently, it is best to restrict the experiment to using one ball-bearing and making repeated measurements of its period of oscillation, and careful measurement of its diameter and the radius of curvature of the mirror, taking careful note of the reading and random uncertainties.

Notes on equipment

Equipment list

- Concave mirror
- Ball bearing
- Spirit level
- Spherometer
- Stopwatch
- Metre rule
- Vernier callipers

Equipment guidance

Mirror:

- Use the spirit level to make sure that the mirror mounting is horizontal.
- When releasing the ball, aim to send it on a path that follows a diameter of the mirror.

Appendix 1: Measuring radius of curvature using a spherometer

A spherometer is an instrument for measuring the radius of curvature of lenses and mirrors. Figure 2 illustrates the device. Here we have three fixed legs located equidistant around a circile of radius r. At the centre of this circle is a moveable leg which can be raised or lowered by means of a screw thread. The distance it is raised or lowered can be measured using the vertical scale and the circular dial scale to an accuracy of 0.01 mm.

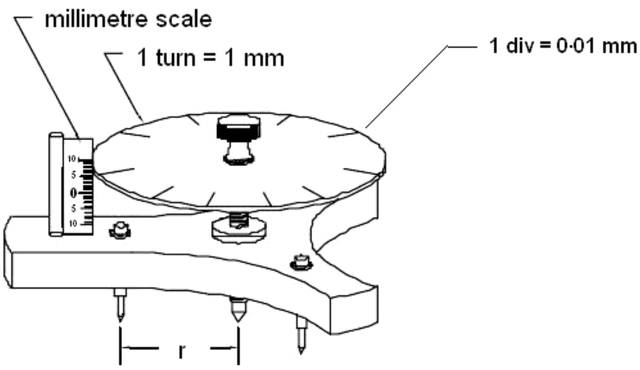

Figure 2

The spherometer is first placed on a plane surface and the centre leg moved until it is just touching the surface to ensure that all four legs are coplanar. The height as read from the vertical and dial scales should be zero but there may be a small discrepancy h_0, which should be noted.

- For a convex surface the centre leg should now be raised sufficiently so that all four legs sit on the surface without rocking.
- For a concave surface the cenrre leg should now be lowered sufficiently so that all four legs sit on the surface without rocking.
- In either case, the height reading h should be read from the vertical and dial scales. This procedure should be repeated several times and a mean value of h obtained.

The distance r between the centre leg and each of the outer legs should be measured with vernier callipers and the mean value of r determined.

The radius of curvature R of the surface can then be determined by the following equation:

$$R = \frac{r^2}{2(h - h_0)} + \frac{(h + h_0)}{2}$$

The derivation of this expression follows ...

Consider the leg arrangement shown in Figure 3. This shows a side view of the spherometer legs resting on a convex surface ADC, of radius of curvature R. The third outer leg is omitted as it would obscure the centre leg. If the legs are projected into the same vertical plane then they are effectively on a circle of radius R and the distances AB and BC are equial to the distance r

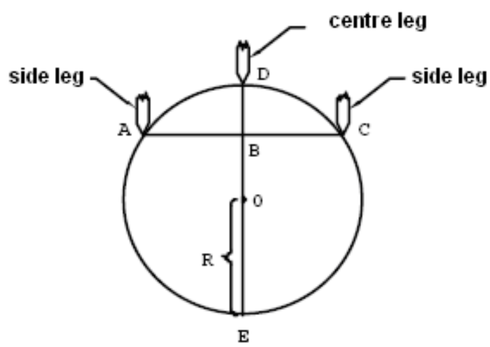

Figure 3

between the centre leg and an outer leg. Similarly, DB is equal to the distance the centre leg was raised from being coplanar with the outer legs to resting on the surface, hence $DB = (h - h_0)$.

By the rule of intersecting chords, $AB \times BC = DB \times BE$, so

$$r^2 = (h - h_0)[2R - (h - h_0)] \Rightarrow r^2 = 2R(h - h_0) - (h - h_0)^2$$

$$\Rightarrow R = \frac{r^2}{2(h - h_0)} + \frac{(h - h_0)}{2}$$

NOTE:

1. Readings on the vertical scale below the 0 mark should be recorded as negative. This will lead to negative values of R for concave surfaces as is the convention.
2. Take care when using the dial scale with concave surfaces – the reading is obtained by counting the divisions backwards (anti-clockwise) from the 0 mark, or taking the scale reading and subtracting it from $1\ \mathrm{mm}$.

Original script: Peter Law

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