# Order of Objects and Flow Algorithm

**Latest Version: Link**

Xihao Song

November 6, 2024

**Abstract**

We examine the random allocation problem and, adhering to the principle of *favoring higher rank*, propose an alternative extension to the probabilistic setting, which does not rely on welfare criteria, to eliminate an unfair scenario where agents who rank objects higher may end up receiving more better objects ex-post also. We introduce the property of *interim favoring support*, which is satisfied by the adaptive Boston Mechanism. Additionally, we propose a new fairness criterion, termed *equal support equal claim*, which further characterizes the adaptive Boston Mechanism.

Furthermore, we incorporate *player 0* into the random allocation framework. This player, who may act as a social planner, manager, or mediator, does not receive any objects but holds incomplete preferences over the resulting allocations based on the full ordinal preference profile including player 0. We present two guiding principles to clarify the conditions under which the social planner's opinion cannot

---

Adam Smith Business School, University of Glasgow, e-mail: xihao.song@glasgow.ac.uk
We are deeply indebted to Herve Moulin and Takashi Hayashi for their continuous guidance throughout this process. We extend our sincere gratitude to Xiangliang Li and Jun Zhang, as well as to the participants of workshops and conferences including the 17th meeting of the Society for Social Choice and Welfare, the 7th World Congress of the Game Theory Society, the Conference on Mechanism and Institution Design, and the 7th International Workshop on Matching Under Preferences, for their valuable comments. Any remaining errors are solely our responsibility.

be dismissed and when an agent's opinion must be respected. The first principle respecting social planner, *Make-Full-Use-Efficiently (MFUE)*, asserts that for any given object, no agent with a stronger bias should receive the object before those with weaker bias are satisfied, which strengthens the notion of ordinal efficiency. The second principle respecting agents, *Equal-Bias-Equal-Treatment (EBET)*, requires that agents in identical bias positions be treated equally. To fulfil these two fairness principles, we develop a method to precisely characterize them.

Finally, we introduce a new efficiency concept, *interim efficiency*, which is stronger than ex-post Pareto efficiency but weaker than ordinal efficiency. We construct the Random Flow mechanism to achieve interim efficiency. Experimental analysis shows that Random Flow results in less envy across preference profiles compared to the Random Priority mechanism.

## 1. Introduction

We must allocate $n$ goods to $n$ agents with ordinal preference and use the lottery to resolve unfairness.

*Extension of favoring higher rank and Adaptive Boston Mechanism.* In many works, people assume that agents are justly entitled to acquire objects based on whether they prefer them more than others and Kojima and Ünver (2014) first introduce *favouring higher rank* to support and characterize *Boston mechanism*. It states for any school and for any two students Ann and Bob if Ann ranks this school higher than Bob. Bob receives the seat of this school only when Ann receives the seat from the school that is better than this school for Ann. Patrick (2018) provides a natural extension to the probabilistic setting called *Interim Favoring Rank* which states for any school, for any two agents Ann and Bob, if Ann ranks this school higher than Bob. Bob receives the seat from this school only when Ann receives 0 from the lower counter set of this school. But it will result in an unfair situation in a probabilistic setting.

| | |
|---|---|
| 1, 2, 3: | $a_1, a_2, a_4, a_3, a_5$ |
| 4,5: | $a_3, a_1, a_2, a_4, a_5$ |

TABLE 1. Preference Profile

In Table 1, *Interim Favoring Rank* stipulates that agents 4 and 5 can receive school $a_4$ (or objects $a_1$ or $a_2$) only if agents 1, 2, and 3 have received nothing from the lower-ranked set associated with $a_4$ (or $a_1$ or $a_2$). However, agents 1, 2, and 3 already have opportunities to access their top two preferred schools ($a_1$, $a_2$), while agents 4 and 5 only have a chance to secure their top-choice school ($a_3$). Under these circumstances, is it fair to grant agents 1, 2, and 3 additional priority over $a_4$?

The issue of fairness has been raised before. As noted by Kojima and Ünver (2014), *favouring higher rank* is typically viewed as a welfare criterion, yet it has also been interpreted as a fairness standard by Ramezanian and Feizi (2021). However, it is not fair enough for a probabilistic setting. In this context, assigning $a_4$ to agents 4 and 5 might be more equitable, as it would ensure a more balanced distribution of opportunities across agents to attend schools they prefer. Importantly, this fairness concern does not arise in deterministic settings, where each agent is allocated only one object.

However, we take fairness criteria and propose the following extension of *favouring higher rank* to probabilistic setting: *interim favouring support*, which states that for any object $a$, for any two agents Ann and Bob, if Bob receives fewer objects preferred to $a$, then Ann receive this object only when Bob is satisfied at object $a$.

It is easy to verify this axiom is stronger than *ordinal efficiency* and most popular mechanisms not satisfy this axiom such as Random Priority, Probabilistic Serial or Boston Mechanism, while *Adaptive Boston Mechanism* satisfies. Bogomolnaia (2015) introduce it to random allocation problem and show it is *ordinally efficient*, *lexi-envy-free* and *lexi-strategy-proof*. The reader can realize the Adaptive Boston Mechanism is also *interim favouring support*.

Considering the characterisation of ABM, we need one more axiom: *equal support equal claim* requires agents to have no incentive to exchange their assignment for this object. This is stronger than *lexi-envy-free*. To notice, Chen, Harless, and Jiao (2023) replace 'support' with 'rank' to characterize the Boston Mechanism.

*Social planner as the agent.* In this extended scenario, we introduce **agent 0**, who plays a role in influencing the allocation but does not receive the school placement for themselves, such as a manager, social planner, or mediator. In this model, we must allocate $n$ goods to $n$ agents with the classic agents' ordinal preferences and agent 0's ordinal preferences. Similarly, we use the lottery to resolve unfairness. From the whole preference profile (including both classic agents and agent 0), agent 0 forms an incomplete preference over the pair of (agent, object) and then forms an incomplete preference over allocations.

It is not only of theoretical interest, but it also makes sense to have agent 0—a neutral party that holds ordinal preferences over objects (in this case, schools)—to ensure fair and balanced allocation in many real-world situations.

In the case of COVID-19 vaccine distribution, vaccine type sometimes took precedence over recipient preferences or characteristics due to logistical and public health priorities. For example, the Pfizer and Moderna mRNA vaccines initially faced distribution challenges due to their need for ultra-cold storage. This limited their availability in rural or low-resource areas that lacked such facilities. In these places, viral vector vaccines, like Johnson & Johnson or AstraZeneca were prioritized for easier distribution and longer shelf life. Here, the choice of vaccine was driven by the logistical priority to maximize the reach and efficacy of the distribution campaign, rather than recipient preference for a particular vaccine type. However, citizens may prefer Pfizer and Moderna to Johnson & Johnson or AstraZeneca and this can not be neglected as well.

The EU's Horizon 2020 program, with a budget of €74.3 billion, exemplifies allocation based on a "priority-based approach" rather than need-based allocation. Key priority areas, such as "Excellent Science" and "Industrial Leadership," are given precedence, ensuring top-tier science and innovation sectors receive funds regardless of specific organizational demand across regions. For example, funds for cutting-edge R&D are directed primarily to projects aligning with EU strategic interests, like climate resilience, energy efficiency, and technological advancements, rather than accommodating regional demand variances.

Examples can also be seen in contexts like college admissions or job placements, where a social planner or manager plays an important role. For instance, in allocating students to colleges, a planner may rank colleges based on quality or societal benefit, considering factors like access, equity, and diversity Salgado-Torres (2013); Xie (2024).

In these scenario, agent 0 does not directly benefit from the allocation but must ensure that the process considers both fairness and societal preferences. This approach can help address imbalances where one party may have stronger preferences or better opportunities, but fairness requires prioritizing others with fewer chances. By introducing agent 0 with their own rankings of the schools, based on overall objectives (such as optimizing student outcomes or ensuring equal access), the allocation becomes more structured and ensures that broader, socially beneficial outcomes are considered alongside individual preferences. In the college context, for example, agent 0 might give priority to students from underrepresented backgrounds for top-tier colleges, even if other students rank those colleges higher because doing so could promote societal equity.

There are many other examples, such as organ allocation with waiting times (Ashlagi, 2024), dairy food for a food bank (Prendergast (2022)), food rescue services (Aydin Alptekinoglu (2023)), and status ranking (Richter and Rubinstein (2024)), among others.

The critical challenge in balancing the social planner's willingness (agent 0) and the individual agents' preferences lies in determining when collective fairness should override personal utility. The social planner's perspective is crucial in situations where equity, broader societal goals, or systemic externalities are at stake, such as in public goods distribution, college admissions, or markets where diversity and fairness are essential. In these cases, prioritizing the planner's preferences ensures a more equitable outcome, as ignoring fairness could lead to unequal access or societal inefficiencies. However, in cases where individual satisfaction and personal utility are the primary concerns, such as when no major fairness issues exist, agents' preferences should take precedence. Overriding individual willingness too often can lead to dissatisfaction, reduced participation, or even inefficiencies. Thus, the solution lies in finding the right balance—using the social planner's influence when fairness and societal goals are at risk while respecting individual preferences when those concerns are minimal.

Imagine the following preference profile

| | |
|---|---|
| 0: | $a_1, a_2, a_3, a_4, a_5, a_6$ |
| 1, 2: | $a_5, a_4, a_2, a_1, a_3, a_6$ |
| 3, 4, 5: | $a_2, a_4, a_1, a_3, a_5, a_6$ |
| 6: | $a_3, a_1, a_2, a_5, a_4, a_6$ |

TABLE 2. Preference Profile

Then Agent 0 will construct the incomplete preference over pairs of (agent, object) after Agent 0 observes the whole preference profile in Table 2.

$$(6, a_1) \succ (3, a_1) \sim (4, a_1) \sim (5, a_1) \succ (1, a_1) \sim (2, a_1) \tag{1}$$

$$(3, a_2) \sim (4, a_2) \sim (5, a_2) \succ (1, a_2) \sim (2, a_2) \sim (6, a_2) \tag{2}$$

$$(6, a_3) \succ (3, a_3) \sim (4, a_3) \sim (5, a_3) \succ (1, a_3) \sim (2, a_3) \tag{3}$$

$$(1, a_4) \sim (2, a_4) \sim (3, a_4) \sim (4, a_4) \sim (5, a_4) \succ (6, a_4) \tag{4}$$

$$(1, a_5) \sim (2, a_5) \succ (6, a_5) \succ (3, a_5) \sim (4, a_5) \succ (5, a_5) \tag{5}$$

$$(1, a_6) \sim (2, a_6) \sim (3, a_6) \sim (4, a_6) \sim (5, a_6) \sim (6, a_6) \tag{6}$$

(7)

Although Agent 0 has a preference over objects, Agent 0 does not receive any object, so how can Agent 0 form a preference over outcomes? We suggest that Agent 0 constructs preferences over outcomes using both first-order stochastic dominance and a lexicographical relation. First, Agent 0 constructs a preference over the allocation of each object based on first-order stochastic dominance. Then, Agent 0 forms a preference over the overall allocation using a lexicographical relation.

For example, if we consider the allocation of object $a_1$ (or $a_3$), would Agent 0 have a preference on how to allocate object $a_1$ among four agents? Yes, Agent 0 might prefer allocating $a_1$ (or $a_3$) to Agent 6. Further, if Agent 0 holds the preference $a_1 \succ a_3$, then Agent 0 may favour the allocation where Agent 6 receives $a_1$ over the one where Agent 6 receives $a_3$. Then, the best allocation for Agent 0 is in Table 3

| | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ |
|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ |
| 2 | 0 | 0 | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ |
| 3 | 0 | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | 0 | 0 |
| 4 | 0 | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | 0 | 0 |
| 5 | 0 | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | 0 | 0 |
| 6 | 1 | 0 | 0 | 0 | 0 | 0 |

TABLE 3. Best for Agent 0

It's easy to see the conflict between Agent 0 and Agents because Agent 6 will trade $a_3$ with $a_1$. Is it justice for Agent 0 to reject this objection? We think Agent 0 shouldn't reject this because of efficiency. Then Agent 1 and 2 may also reject the allocation of the object $a_4$, would Agent 0 approve this objection? Agent 0 may not reject this because Agent 1 and 2 prefer the object $a_5$ to $a_4$. In contrast, Agent 0 prefers $a_4$ to $a_5$, in other words, Agent 1 and 2 have different concerns about $a_4$ and $a_5$ and these concerns are not accepted by Agent 0 according to his preference. Then Agent 0 may still give $a_4$ to Agents 3,4 and 5. Then the allocation after negotiation is more like the Table 4.

| | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ |
|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ |
| 2 | 0 | 0 | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ |
| 3 | $\frac{1}{3}$ | $\frac{1}{3}$ | 0 | $\frac{1}{3}$ | 0 | 0 |
| 4 | $\frac{1}{3}$ | $\frac{1}{3}$ | 0 | $\frac{1}{3}$ | 0 | 0 |
| 5 | $\frac{1}{3}$ | $\frac{1}{3}$ | 0 | $\frac{1}{3}$ | 0 | 0 |
| 6 | 0 | 0 | 1 | 0 | 0 | 0 |

TABLE 4. Allocation after negotiation

Then we have the following two principles for the ideal allocation in this model, the first one is called *Make-Full-Use-Efficiently*:

> the object should be allocated to the agents who value them in the much more 'correct' position corresponding to Agent 0's preference.

When we refer to an agent who values the object in the much more 'correct' position, we're describing an agent who violates the order of objects less and also potentially receives fewer objects preferred to this object. Also, this is stronger than ordinal efficiency. Similarly, it is natural to require the second property (called *Equal bias equal treatment*):

> those agents should receive an equal share for fairness.

Therefore, we have three possible outcomes in Table 11.

| | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ |
|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ |
| 2 | 0 | 0 | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ |
| 3 | $\frac{1}{3}$ | $\frac{1}{3}$ | 0 | $\frac{1}{3}$ | 0 | 0 |
| 4 | $\frac{1}{3}$ | $\frac{1}{3}$ | 0 | $\frac{1}{3}$ | 0 | 0 |
| 5 | $\frac{1}{3}$ | $\frac{1}{3}$ | 0 | $\frac{1}{3}$ | 0 | 0 |
| 6 | 0 | 0 | 1 | 0 | 0 | 0 |

TABLE 5. MFUE+EBET

| | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ |
|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ |
| 2 | 0 | 0 | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ |
| 3 | 0 | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | 0 | 0 |
| 4 | 0 | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | 0 | 0 |
| 5 | 0 | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | 0 | 0 |
| 6 | 1 | 0 | 0 | 0 | 0 | 0 |

TABLE 6. Ob-eff + fairness

| | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ |
|---|---|---|---|---|---|---|
| 1 | $\frac{1}{5}$ | 0 | $\frac{1}{30}$ | $\frac{1}{10}$ | $\frac{1}{2}$ | $\frac{1}{6}$ |
| 2 | $\frac{1}{5}$ | 0 | $\frac{1}{30}$ | $\frac{1}{10}$ | $\frac{1}{2}$ | $\frac{1}{6}$ |
| 3 | $\frac{1}{5}$ | $\frac{1}{3}$ | $\frac{1}{30}$ | $\frac{4}{15}$ | 0 | $\frac{1}{6}$ |
| 4 | $\frac{1}{5}$ | $\frac{1}{3}$ | $\frac{1}{30}$ | $\frac{4}{15}$ | 0 | $\frac{1}{6}$ |
| 5 | $\frac{1}{5}$ | $\frac{1}{3}$ | $\frac{1}{30}$ | $\frac{4}{15}$ | 0 | $\frac{1}{6}$ |
| 6 | 0 | 0 | $\frac{5}{6}$ | 0 | 0 | $\frac{1}{6}$ |

TABLE 7. PE + fairness

| | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ |
|---|---|---|---|---|---|---|
| 1 | $\frac{3}{20}$ | 0 | $\frac{1}{30}$ | $\frac{3}{20}$ | $\frac{1}{2}$ | $\frac{1}{6}$ |
| 2 | $\frac{3}{20}$ | 0 | $\frac{1}{30}$ | $\frac{3}{20}$ | $\frac{1}{2}$ | $\frac{1}{6}$ |
| 3 | $\frac{7}{30}$ | $\frac{1}{3}$ | $\frac{1}{30}$ | $\frac{7}{30}$ | 0 | $\frac{1}{6}$ |
| 4 | $\frac{7}{30}$ | $\frac{1}{3}$ | $\frac{1}{30}$ | $\frac{7}{30}$ | 0 | $\frac{1}{6}$ |
| 5 | $\frac{7}{30}$ | $\frac{1}{3}$ | $\frac{1}{30}$ | $\frac{7}{30}$ | 0 | $\frac{1}{6}$ |
| 6 | 0 | 0 | $\frac{5}{6}$ | 0 | 0 | $\frac{1}{6}$ |

TABLE 8. Truthful Mechanism

| | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ |
|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | $\frac{1}{5}$ | $\frac{1}{2}$ | $\frac{3}{10}$ |
| 2 | 0 | 0 | 0 | $\frac{1}{5}$ | $\frac{1}{2}$ | $\frac{3}{10}$ |
| 3 | $\frac{1}{3}$ | $\frac{1}{3}$ | 0 | $\frac{1}{5}$ | 0 | $\frac{2}{15}$ |
| 4 | $\frac{1}{3}$ | $\frac{1}{3}$ | 0 | $\frac{1}{5}$ | 0 | $\frac{2}{15}$ |
| 5 | $\frac{1}{3}$ | $\frac{1}{3}$ | 0 | $\frac{1}{5}$ | 0 | $\frac{2}{15}$ |
| 6 | 0 | 0 | 1 | 0 | 0 | 0 |

TABLE 9. Favoring Rank + fairness

| | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ |
|---|---|---|---|---|---|---|
| 1 | $\frac{1}{5}$ | 0 | 0 | $\frac{1}{5}$ | $\frac{1}{2}$ | $\frac{1}{10}$ |
| 2 | $\frac{1}{5}$ | 0 | 0 | $\frac{1}{5}$ | $\frac{1}{2}$ | $\frac{1}{10}$ |
| 3 | $\frac{1}{5}$ | $\frac{1}{3}$ | 0 | $\frac{1}{5}$ | 0 | $\frac{4}{15}$ |
| 4 | $\frac{1}{5}$ | $\frac{1}{3}$ | 0 | $\frac{1}{5}$ | 0 | $\frac{4}{15}$ |
| 5 | $\frac{1}{5}$ | $\frac{1}{3}$ | 0 | $\frac{1}{5}$ | 0 | $\frac{4}{15}$ |
| 6 | 0 | 0 | 1 | 0 | 0 | 0 |

TABLE 10. Favoring Support + fairness

TABLE 11. Comparison

In this context, an efficient and fair allocation method must consider both Agents and Agent 0 (Table 5). Now, we propose a simple method to achieve this outcome. Following Agent 0's preference, we equally allocate to the agents who prefer the most according to their capacity. Formally, it is called Flow algorithm:

(a) In round 1, we require the object in the first position of Agent 0's preference, $\pi(1)$, to appear in the market, then equally allocate them to agents who put them on the top of the total objects. If not, we add this object in the next position.

(b) In round $k$, we have the left objects in the previous round and the objects in the $k-th$ position, $\pi(k)$. Then we allocate the objects in $\pi(k)$ to agents who put them at the top of the remaining objects equally considering their capacity. We keep this process until no one puts the object in $\pi(k)$ at the top of the remaining objects or $\pi(k)$ is exhausted.

It will return the probabilistic allocation directly and use the lottery between agents during allocation.

*Interim efficiency and Random Flow.* In literature, there are two dominant methods to solve the general problem (without perishability): Random Priority (RP) (Abdulkadiroğlu and Sönmez (1998)) [1], and Probabilistic Serial (PS) (Bogomolnaia and Moulin (2001)).

A sequence of agents (termed *priority*) serves as a natural tool to solve this problem. The manager sequentially asks agents, based on an exogenous priority, to choose their most preferred object from the remaining ones. This mechanism is renowned for its efficiency and incentive compatibility, but it falls short in terms of fairness. To address this, *random priority* is employed: the manager randomly determines an ordering and then queries agents to select their best object from what remains. [2]

However, Bogomolnaia and Moulin (2001) shows RP lacks efficiency, namely it is not *ordinally efficient*. An assignment is *ordinally efficient* for some problems if there is no probabilistic Pareto improvement. Therefore they construct the *probabilistic serial*. In this method, the manager directly allocates a divisible probability weight. Agents simultaneously 'eat' the probability weight of their most preferred available object at a uniform rate. Once an object is fully 'eaten' by some agents, they move on to their next most preferred yet uneaten object. This process continues until all objects are completely allocated.

One advantage of *ordinal efficiency* is that it guarantees any decomposition of random allocation is the convex combination of efficient deterministic allocations while the efficiency of RP only guarantees there exists a convex combination of efficient deterministic allocations called *ex-post Pareto efficiency*.

[1] Also known as random serial dictatorship in literature.

[2] Recently, RP is shown the unique rule satisfies *symmetry*,*ex-post Pareto efficient*, and *obvious strategy-proof*.

We realize there is a natural extension of *ex-post Pareto efficiency* such that the random allocation can decompose into a convex combination of probabilistic ordinally efficient allocations. We call it *interim efficiency*. The *interim efficiency* is logically squeezed between ordinal efficiency and ex-post Pareto Efficiency: every ordinally efficient allocation is *'interim efficient'* and every *'interim efficient'* allocation is Ex-post Pareto efficient (the converse is not true). Unfortunately, Random Priority is not *interim efficient*.

We want to propose another more efficient rule that is also easy to practice in reality. Imprecisely, we randomly generate an order of objects (permutation), $\pi$, then run the Flow algorithm. We can show this method is *interim efficient* in Table 12. One of the main assets of this algorithm is that everyone can quickly understand how the method works as well as RP.

| | Efficiency | Fairness | Incentive Compatibility |
|---|---|---|---|
| RF | Interim Efficient | weakly sd Envy-Free | weakly sd Strategy-Proof |
| RP | Ex-post Pareto Efficient | weakly sd Envy-Free | sd Strategy-Proof |

TABLE 12. comparison Between RP and RF

We provide an experimental analysis among existing dominant mechanisms and RF. In $4 \times 4$ case, we observed RF generates no-envy in more preference profiles than RP: RP generates sd-envy-free allocation in 36% of preference profiles while RF generates sd-envy-free allocation in 48% of preference profiles. It suggests we can design an easy algorithm based on RF that is superior to RP with efficiency and fairness.

This paper is structured as follows: Section 2 presents the Preliminaries. Section 3 describes the 'Adaptive Boston Mechanism' and characterization. Section 4 introduces the order of objects and the new desired properties, the new method, and the characterization. Section 5 introduces a new notion of efficiency *'interim efficiency'* and Section 6 introduces the Random Flow algorithm with its necessary conditions.

### 1.1. Literature Review

This project contributes to different questions.

*Favor higher ranks and Adaptive Boston Mechanism.* Firstly, it proposes an extension of favoring higher rank (Kojima and Ünver (2014)) by fairness criteria instead of welfare criteria (such as Patrick (2018), Ramezanian and Feizi (2021), etc.) It's not hard to imagine Boston mechanism will not satisfy this property (Chen, Harless, and Jiao (2023)) but the Adaptive Boston Mechanism will do. Adaptive Boston Mechanism is well-known in school choice and has better performance in efficiency and incentive compatibility than the naive Boston Mechanism, as illustrated by Mennle and Seuken (2014) and Mennle and Seuken (2021). In this paper, one important classification between the Boston Mechanism and the adaptive Boston Mechanism is the fundamental difference between welfare and fairness.

To characterize the adaptive Boston Mechanism, we propose another fairness property that is stronger than Lexi-envy-freeness. This also replies to the question in Bogomolnaia (2015): which both ABM is not single out from mechanisms that are Lexi-efficiency, Lexi-envy-freeness, and Lexi-strategy-proofness.

*Mechanism Design for the market with the order of objects.* It is not the first work considering the rank of objects or the order of objects in the market. It is worth noting two recent works. Liu and Zeng (2019) provides the algorithm on restricted tier domain, simply the preference is consistent with public rank. While we do not impose any restriction on the preference domain. Also, Harless (2019) characterizes all *sd-efficient* algorithms using the *order-claim-algorithm*. Although efficiency is not the only focus of this paper, there is the same spirit between Harless (2019) and this paper: generating the order of objects and allocating them efficiently. However, we care about why there is an inconsistency between preference and initial order not only efficiency, and how to allocate the object given this inconsistency.

The adaptation of stability in allocation problems always leads to nice axioms in literature (see Gale and Shapley (1962), Abdulkadiroğlu and Sönmez (2003b)): an object should not be given to an agent with a lower priority when it is desired by an agent with a higher priority to it. The natural generalization of stability to probabilistic setting is introduced by Roth, Rothblum, and Vande Vate (1993) and analyzed further by Kesten and Ünver (2015) (p552.):

> ... this notion stipulates that, being the higher priority student, student $i$ should be granted all the enrollment chance at school $c$, should he so desire, before student $j$ is given any chance at this school.

The adaption of stability in allocation problem is called *favoring higher rank* by Kojima and Ünver (2014) or *ex-ante stability* by Han (2023). In this work, we introduce the order of objects and give another reasonable way to consider priority in the allocation problem.

*Refinement of Ex-post Pareto Efficiency.* Lastly, it is well-known that ordinal efficiency is stronger than ex-post efficiency and RP is not ordinally efficient. Then it is always interesting to check the boundary of efficiency for RP and think it happens. Interim efficiency requires that the random allocation can be decomposed (if there is one) into a convex combination of ordinally efficient random allocations. It is exactly between ordinal efficiency and ex-post efficiency and we show RP does not satisfy this property. [3]

## 2. Preliminary

Consider a classic assignment problem with indivisible goods. For any positive integer $x$, define $[x] = \{1, 2, ..., x\}$. Let $N = [n]$ denote a set of agents, and $A = [n]$ denote a set of goods. The capacity of each agent and each object is 1. We consider the set of strict preferences $\mathcal{R}$ on $A$, the representative element is $R$. We use $R_N$ to represent the preference profile.

A random assignment is a bistochastic $P = [p_{ia}]_{i \in N, a \in A}$.[4] The set of random assignments is denoted $\mathcal{P}$. We use $P_i$ to represent the allocation of agent $i$. A random assignment rule is a mapping $f : \mathcal{R}^N \to \mathcal{P}$. We use $f_i(R_N)$ to represent the allocation/probability that agent $i$ receives under the assignment rule $f$ and use $f_{ia}(R_N)$ to represent the allocation/probability that agent $i$ receives object $a$ under rule $f$ and preference profile $R_N$.

We define the upper contour set of $R_i$ at an object $a \in A$ as $U(R_i; a) = \{x : xR_ia\}$ and weak upper contour set of $R_i$ at an object $a \in A$ as $\overline{U}(R_i; a) = U(R_i; a) \cup \{a\}$.

For all $i$, given a preference $R_i$ on $A$, we call a partial ordering of the set $\Delta(A)$ the *stochastic dominance* relation associated with $R_i$ and denoted $R_i^{sd}$ if $\forall P_i, P_i' \in \Delta(A)$ we have

$$P_i R_i^{sd} P_i' \Leftrightarrow \sum_{x \in \overline{U}(R;a)} P_i \geq \sum_{x \in \overline{U}(R;a)} P_i', \forall a \in A.$$

Given a preference $R_i$ on $A$, $\forall P_i, P_i' \in \Delta(A)$, we say $P_i'$ is *stochastically dominated* by $P_i$ for agent $i$ if we have $P_i R_i^{sd} P_i'$ and $P_i \neq P_i'$.

[3]We think there is one open and interesting question: Do interim efficiency and robust ex-post efficiency imply ordinal efficiency? Robust ex-post efficiency see Aziz et al. (2015) and Ramezanian and Feizi (2022). Abdulkadiroğlu and Sönmez (2003a) also gives some thoughts on why ex-post efficiency is not ordinal efficiency: a random assignment is ordinally efficient if and only if for any given feasible support, each of its subsets is undominated.

[4]That is, $P \in [0, 1]^{N \times A}$ and for each $i \in N$ and $a \in A$, $\sum_{b \in A} P_{ib} = 1$ and $\sum_{j \in N} p_{ja} = 1$.

We define $P$ as *ordinally efficient* if $P_i$ is not stochastically dominated for all $i$. A random assignment rule $f$ is *ordinally efficient* if, for all $R \in \mathcal{R}^n$, $f(R)$ is ordinally efficient.

## 3. Extension of favoring higher rank

We need one definition to introduce the property. Whenever $R_N$, $P$, $i$ and $a$, we define $Z(i; a; P) = \#\{b : b \in \overline{U}(R_i; a), P_{ib} > 0\}$ as the index of $a$ to indicates the number of objects preferred to $a$ with a positive probability for the agent $i$.

PROPERTY 1. *Interim Favoring Support*

*For all $R_N$, all $a$, all $i$, if $P_{ia} > 0$, then* $\sum_{x \in \overline{U}(R_j;a)} P_{jx} = 1$ *for all $j$ that* $Z(j; a; P) < Z(i; a; P)$.

Property 1 states that whenever an agent $i$ receives a positive share of some object $a$, then all agents who receive fewer objects preferred to $a$ than agent $i$ should be satisfied at object $a$. Moreover, it is stronger than ordinal efficiency.

Different from Kojima and Ünver (2014) and Patrick (2018), this property entitles the right of the object to agents based on what they will receive instead of how they value the object. In the previous example, if one agent (he) ranks one object $a$higher than another (she) but he already receives more objects (possibly) than another, *interim favoring support* will allocate $a$ to her while *interim favoring rank* will allocate $a$ to him. To notice, Chen, Harless, and Jiao (2023) also replaces $Z(i; a; P)$ with $r(i; a)$, rank of $a$ for agent $i$ in $R_i$, in their characterization of Boston Mechanism.

PROPOSITION 1. *The property 1 implies Ordinal efficiency.*

Now, we start with the Adaptive Boston Mechanism in a random assignment problem. To provide the formal definition, we need to define: whenever $B \subseteq A, N' \subseteq N, a \in B, M(a; B; N') \equiv \{i \in N' : aR_i b, \forall b \in B\}$ and $m(a; B; N') = \#M(a; B; N')$. In words, the $M(a; B; N')$ is the set of agents in $N'$ who put $a$ at the top of $B$ and the set of $M(a; B; N')$ is a partition of $N'$.

Given a preference profile $R_N$, the Adaptive Boston Mechanism proceeds sequentially. Let $A^0 = A, N^0 = N$, let $C = [1]^N$ be the capacity of agents, $Z = [1]^A$ be the capacity of objects.

(a) In the first period, for all $a$ if $M(a; A^0; N^0) \neq \emptyset$, we fully allocate $a$ to $M(a; A^0; N^0)$ and every agent receives $sh_{ia} = \frac{1}{m(a;A^0;N^0)}$. Then we update capacity $Z^1$ and $C^1$, the remaining objects $A^1$, and the agents with positive capacity $N^1$.

(b) For each $k$ period, for all $a \in A^{k-1}$ if $M(a; A^{k-1}; N^{k-1}) \neq \emptyset$, we allocate $a$ to agents $M(a; A^{k-1}; N^{k-1})$ and every agent receives, for some $e$:

$$sh_{ia} = c_i^{k-1} \wedge e \quad s.t. \quad \sum_{i \in M(a;A^{k-1};N^{k-1})} sh_{ia} = z_a^{k-1} \wedge \sum_{i \in M(a;A^{k-1};N^{k-1})} c_i^{k-1}$$

Then we update capacity $Z^k$ and $C^k$, the remaining objects $A^k$, and the agents with positive capacity $N^k$.

This algorithm will finish in finite periods at most $|A|$ and produce an allocation matrix $sh$.

Now we state the second fairness property stronger than Lexi-envy-free: requires such agents to have no incentive to exchange their assignment for this object also.

Property 2. *Equal Support Equal Claim*

*For all $R_N$, all $a$, for all $i$, $j$ s.t. $Z(i;a;P) = Z(j;a;P)$, if $P_{ia} > P_{ja} > 0$ then* $\sum_{x \in \overline{U}(R_j;a)} P_{jx} = 1$.

Given the stick preference $R_i : a_1 R_i a_2 \ldots R_i a_n$, we define lexicographic preference $R_i^{lex}$ over all probability distributions $\Delta(A)$: for all $p, q \in \Delta(A)$, then we say $p R_i^{lex} q$ as long as there is $j \in \{1, ..., n\}$ such that $p_{a_j} > q_{a_j}$, while $p_{a_k} = q_{a_k}$ for all $k < j$.

Given an allocation $P$ and preference profile $R_N$, we say it is *lexi-envy-free*, if for any agents $i$ and $j$, we have $P_i R_i^{lex} P_j$ and $P_j R_j^{lex} P_i$.

Proposition 2. *Property 2 implies lexi-envy-free.*

Proposition 3. *A mechanism is the Adaptive Boston mechanism in a random assignment problem if and only if it satisfies interim favoring support and equal support equal claim.*

Then we show interim favoring support and equal support equal claim are independent. Moreover, equal claim equal support and ordinal efficiency are not sufficient to obtain interim favoring support. Interim favoring support and Lexi-envy-free are not sufficient to obtain equal claim equal support.

Proposition 4.

(a) *Lexi-envy-free and interim favoring support do not imply equal support equal claim.*

(b) *Ordinally efficiency and equal support equal claim do not imply Property interim favoring support.*

## 4. Agent 0's preference and principles

In this section, we consider the main question of this paper: how to allocate the object if there is Agent 0. We denote $\pi \in \Pi$ as Agent 0's strict preference and We denote $\pi_a$ as the position of $a$ in $\pi$, that is $\#\{x : x\pi a\} + 1$. Then we say $a$ is in prior position than $b$ in $\pi$ if $\pi_a < \pi_b$.

First, let's define Agent 0's preference over allocation. We denote

$$wr(a;P;M) = (\sum_{i:r(i;a;P)=1} M_{ia}, \dots, \sum_{i:r(i;a;P)=n} M_{ia})$$

Given the allocation $M$ and $M'$, for any $a$ define $wr(a;P;M) \mid \triangleright\, wr(a;P;M')$ if $\sum_{j=1}^{k} wr(a;P;j) \geq \sum_{j=1}^{k} wr(a;P;j)$ for all $k = 1, 2, \dots, n$. We say $M \mid \triangleright_{lex} M'$ if $wr(a;P;M) \mid \triangleright\, wr(a;P;M')$ for some $a$ and $wr(b;P;M) = wr(b;P;M')$ for all $b$ that $\pi_b < \pi_a$. We say $M$ is ob-efficient if there is no $M'$ s.t. $M' \mid \triangleright_{lex} M$.

LEMMA 1. *ob-efficiency and ordinal efficiency are not compatible.*

Because ordinal efficiency and ob-efficiency are not compatible, we consider the properties mentioned in the Introduction. Make-Full-Use-Efficiently (MFUE) says that the object should be allocated to the agents who value them in the much more 'correct' position corresponding to Agent 0's preference. Equal-Bias-Equal-Treatment (EBET) says that those agents should receive an equal share for fairness.

Now we formally state those two properties. Whenever $R_N$, and $a \in A$ we define $C(i;a) = \arg\max\{\pi_x : x \in \overline{U}(R_i;a)\}$ and $\overline{*}(i;a) = \arg\min\{\pi(i;x) : x \in C(i;a)\}$ be the peak before $a$. Denote $\overline{*}(i;A) = \{b : \exists a \in A, b = \overline{*}(i;a)\}$ be the set of peaks of agent $i$.

EXAMPLE 1. *In the initial example Table 2, let's focus on Agent 6:*

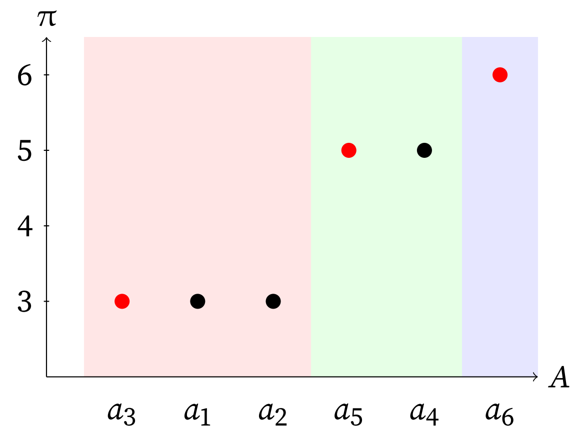


FIGURE 1. Peaks of agent $i$ in Example 1

**Note 1:** Red points indicate peaks. x-axis represents objects. y-axis represents the $\pi_a$.

*We have*

$$\overline{*}(6;a_1) = \overline{*}(6;a_3) = \overline{*}(6;a_3) = a_3,\ \overline{*}(6;a_4) = \overline{*}(6;a_5) = a_5,\ \overline{*}(6;a_6) = a_6.$$

*Then* $\overline{*}(6;A) = \{a_3, a_5, a_6\}$ *and* $a_6$ *is the last peak in Figure 1.*

Whenever $R_N$ and $P$, for all $i$ and $a \in A$, we define $\hat{U}(R_i;a) = \{b : \overline{*}(i;a) R_i b R_i a\} \cup \{a\} \cup \{\overline{*}(i;a)\}$. Then we define $\hat{Z}(i;a;P) = \#\{b : b \in \hat{U}(R_i;a), P_{ib} > 0\}$. For example:

EXAMPLE 2. *(Example 1 Cont'd.)*

*Now assume* $P_6 = (\frac{1}{5}, \frac{1}{5}, 0, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$. *Then we have:*

$\hat{Z}(6;a_1;P) = 1$, $\hat{Z}(6;a_2;P) = 2$, $\hat{Z}(6;a_3;P) = 0$, $\hat{Z}(6;a_4;P) = 2$, $\hat{Z}(6;a_5;P) = 1$, $\hat{Z}(6;a_6;P) = 1$.

Whenever $\succ$, $P$, $a \in A$, $i, j \in N$, and $P_{ia} > 0$ we define

(a) $i \xrightarrow{a} j$ if either $\pi_{\overline{*}(i;a)} < \pi_{\overline{*}(j;a)}$ or $\pi_{\overline{*}(i;a)} = \pi_{\overline{*}(j;a)}$ with $\hat{Z}(i;a;p) < \hat{Z}(j;a;p)$.

(b) $i \sim j$ if $\pi_{\overline{*}(i;a)} = \pi_{\overline{*}(j;a)}$ and $\hat{Z}(i;a;p) = \hat{Z}(j;a;p)$

Now we are ready to state the following Properties.

PROPERTY* 1. *Make-Full-Use-Efficiently (MFUE)*

*For all* $R_N$, *all* $\pi$, *all* $a$, *all* $i$, *if* $P_{ia} > 0$, *then* $\sum_{x \in \overline{U}(R_i;a)} P_{jx} = 1$ *for all* $j$ *that* $j \xrightarrow{a} i$.

PROPERTY* 2. *Equal bias equal treatment (EBET)*

*For all* $R_N$, *all* $\pi$, *all* $a$, *all* $i, j$ *s.t.* $i \sim j$, *if* $P_{ia} > P_{ja} > 0$ *then* $\sum_{x \in \overline{U}(R_j;a)} P_{jx} = 1$.

PROPOSITION 5. *MFUE implies ordinal efficiency.*

### 4.1. Flow Algorithm

Before defining the new method, we define a variant of the Adaptive Boston Mechanism which allocates the subset of objects to agents with consideration of full preference. Now given $R_N, P, B, A, N$, whenever $R_N$ is preference profile, $P$ is the matrix, $B$ is to-assign-objects, $A$ is set of objects, $N$ is set of agents, we define $BM(B; A; N)$ as the constraint adaptive Boston mechanism in the following sequential procedure. Let the capacity of objects be $Z^0 = [1]^A - [\sum_{i \in N} P^0_{ia}]_{a \in A}$ the capacity of agents be $C^0 = [1]^N - [\sum_{a \in A} P_{ia}]_{i \in N}$.

(a) In the first period, for all $a \in B$ if $M(a; A^0; N^0) \neq \emptyset$, we fully allocate $a$ to $M(a; A^0; N^0)$ and every agent receives $sh_{ia} = \frac{1}{m(a;A^0;N^0)}$. Then we update capacity $Z^1$ and $C^1$, the remaining objects $A^1$ and $B^1$, and the agents with positive capacity $N^1$.

(b) For each $k$ period, for all $a \in B^{k-1}$ if $M(a; A^{k-1}; N^{k-1}) \neq \emptyset$, we allocate $a$ to agents $M(a; A^{k-1}; N^{k-1})$ and every agent receives, for some $e$:

$$sh_{ia} = c_i \wedge e \quad s.t. \quad \sum_{i \in M(a;A^{k-1};N^{k-1})} sh_{ia} = z_a \wedge \sum_{i \in M(a;A^{k-1};N^{k-1})} c_i^{k-1}$$

Then we update capacity $Z^k$ and $C^k$, the remaining objects $A^k$ and $B^{k-1}$, and the agents with positive capacity $N^k$.

This algorithm will finish in finite periods at most $|B|$ and produce an allocation matrix $sh$. The Constraint Adaptive Boston Mechanism is the variant Adaptive Boston Mechanism when we allocate a set of objects $B$ by considering full preference. To notice, the object in $B$ is not necessary to be fully allocated because it may not be at the top of unassigned objects in $A$ for all agent $i$.

Now, we construct an algorithm to satisfy MUFE and EBET. Given a preference profile $R_N$ and $\pi$. The algorithm is defined by the sequential procedure. Let $A^0 = A, N^0 = N, \pi^0 = \pi, C^0 = [1]^N, Z^0 = [1]^A$,.

(a) In first period, we run $BM(\pi(1); A^0; N^0)$, then update $\pi^1$ as the set of unassigned objects in $\pi(1)$, $A^1$ as the set of unassigned objects in $A^0$, $N^1$ as the set of agents who do not approach the capacity, $C^1$ as the capacity of agents, $Z^1$ as the capacity of objects.

(b) For each k period, denote $\pi(k) = \pi(k) \cup \pi^{k-1}$ we run $BM(\pi(k); A^{k-1}; N^{k-1})$, then update $\pi^{k'}, A^k, N^k, C^k, Z^k$.

This algorithm is denoted by $F^{\pi}$ and will finish in finite periods. The method is simple, it allocates the object set by set according to the $\pi$, for each set we run the constraint Adaptive Boston Mechanism until it is fully allocated or no agent prefers it the most. Now we use the initial example in Table 2 to illustrate this method.

EXAMPLE 3. *We consider $R_N$ and $\pi$ in Table 2, then the method works sequentially:*

(a) *In step 1, allocate the object in $\pi(1) = \{a_1\}$. No one prefers $a_1$ to the rest, then $a_1$ passes to the next period.*

(b) *In step 2, allocate the object in $\pi(2) \cup \{a_1\}$.*

- *Agent 3,4,5 prefer $a_2$ and receive $\frac{1}{3}$ each.*
- *After this, no one prefers $a_1$ to the rest, then $a_1$ passes to the next period.*

(c) *In step 3, allocate the object in $\pi(3) \cup \{a_1\}$.*

- *Agent 6 prefers $a_3$ and receive 1.*
- *After this, no one prefers $a_1$ to the rest, then $a_1$ passes to the next period.*

(d) *In step 4, allocate the object in $\pi(4) \cup \{a_1\}$.*

- *Agent 3,4,5 prefer $a_4$ and receive $\frac{1}{3}$ each.*
- *After Agent 3,4,5 prefer $a_1$ and receive $\frac{1}{3}$ each.*

(e) *In step 5, allocate the object in $\pi(5)$. Agent 1,2 prefer $a_5$ and receive $\frac{1}{2}$ each.*

(f) *In step 6, allocate the object in $\pi(6)$. Agent 1,2 prefer $a_6$ and receive $\frac{1}{2}$ each.*

*Then the allocation is in Table 5 compared to the existing algorithms.*

PROPOSITION 6. *A mechanism is $F^{\pi}$ if and only if it satisfies MFUE and EBET.*

## 5. Interim Efficiency

In random allocation, a randomization device is crucial. Ordinal efficiency (OE) is important because it ensures that every possible way of breaking down the allocations is Pareto efficient. In contrast, ex-post Pareto efficiency (EPPE) only requires that an efficient randomization device exists. Now, we consider an intermediate notion, interim efficiency (IE), which allows for more efficient randomization devices.

Given a preference profile $R_N$, let $\mathcal{D}^R$ denote the set of **deterministic efficient assignments**, and $\mathcal{P}^R$ denote the set of **probabilistic ordinally efficient assignments**. The union $\mathcal{D}^R \cup \mathcal{P}^R$ represents the entire set of efficient assignments.

A random assignment is *ex-post Pareto efficient* if it can be decomposed into a convex combination of deterministic efficient assignments, i.e., $\sum_{k=1}^{K} \lambda_k D_k^R$ with $\lambda_k > 0$ and $D_k^R \in \mathcal{D}^R$ for all $k$.

A random assignment is *'Interim Efficient'* if it can be decomposed into a convex combination of probabilistic ordinally efficient assignments, i.e., $\sum_{k=1}^{K} \lambda_k P_k^R$ with $\lambda_k > 0$ and $P_k^R \in \mathcal{P}^R$ for all $k$.

We will demonstrate that these two notions are distinct, specifically, that *'Interim Efficient'* refines *ex-post Pareto efficiency* but is less stringent than *ordinal efficiency*.

PROPOSITION 7.

(a) *Interim Efficiency implies ex-post pareto efficiency, but the converse is not true.*

(b) *Ordinal Efficiency implies Interim Efficiency, but the converse is not true.*

(c) *Random Priority is Ex-post Pareto Efficient but not Interim Efficient.*

Proposition 7 explains that interim efficiency (IE) is a more refined concept than ex-post Pareto efficiency (EPPE) but is less demanding than ordinal efficiency (OE). IE is significant because it permits more types of efficient randomization devices, although it is less strict than OE, which demands efficiency in all possible deterministic assignments.

## 6. Random Flow

In this part, we consider methods that are easy to implement in practice, such as Random Priority, and examine those that are superior to Random Priority in terms of efficiency (based on axioms) and fairness (based on experiments).

We randomly arrange a series of objects and present them sequentially to the agents. In each period, we assign objects by the constraint Adaptive Boston Mechanism. If the object is not assigned or fully assigned, it passes to the next period. Once an agent gets proportion 1 at all, they exit the process. It will end in $|A|$ periods. This method results in a random allocation directly.[5]

[5] The full description is in the Appendix.

This is the opposite of Random Priority. We call it Random Flow and we show Random Flow has better performance in efficiency, called interim efficient (explained in the next part), which Random Priority does not. Moreover, Random Flow is at least neutral, weakly strategy-proof, and weakly envy-free.

We denote $\Pi$ as the set of permutations of $A$. Given $\pi \in \Pi$, we denote $F^{\pi}$ as the flow algorithm with order $\pi$. Then a mechanism is a Random Flow mechanism if for any $R_N$ it selects the allocation:

$$\frac{1}{|\Pi|} \sum_{\pi \in \Pi} F^{\pi}(R_N)$$

**Remark**. In practical scenarios where many agents require a limited number of valuable items, the Random Flow (RF) mechanism demonstrates superior computational efficiency compared to the Random Priority method. This efficiency stems from randomizing the order of objects rather than the order of agents.

### 6.1. Necessary condition for Random Flow

*sd Envy-freeness.* A random assignment rule $f$ is *sd-Envy-Free* if $\forall R_N, \forall i \in N$ we have $f_i(R) R_i^{sd} f_j(R), \forall j \neq i$. A random assignment rule is *weakly sd-Envy-Free* if no agent strictly prefers someone else's allocation to him or her, that is $f_j(R_N) R_i^{sd} f_i(R_N)$ indicates $f_j(R_N) = f_i(R_N), \forall j \neq i, \forall i \in N$.

*sd strategy-proofness.* A random assignment rule is *sd-strategy-proof* if $\forall i \in N, \forall (R_i)_{i \in N}$ and $\forall R' \in \mathcal{R}$, $f_i(R) R_i^{sd} f_i(R_i', R_{-i})$. A random assignment rule is *weakly sd-strategy-proof* if an agent cannot obtain an allocation that strictly stochastic dominates to a true allocation by telling a lie, that is $f_j(R_i', R_{-i}) R_i^{sd} f_i(R_N)$ indicates $f_i(R_i', R_{-i}) = f_i(R_N)$, $\forall i \in N, \forall R_i' \in \mathcal{R}$.

Proposition 8. *RF is interim efficient, weakly envy-free, and weakly strategyproof.*

### 6.2. Experimental testing of fairness

For the random allocation problem, we consider two mechanisms: Random Priority (RP) and Random Flow (RF), and focus on the property sd-envy-free and ask the question of which algorithm will generate 'no-envy' allocation more often. We focus on the case $n = 4$ and all preference profiles. We began by selecting unique preference profiles and eliminating permutations of rows and columns. Each method was then applied to these profiles to determine allocations.

**Observation: RF generates more sd envy-free allocations than RP.**

Figure 2 shows RP generates sd-envy-free allocation in 36% of preference profiles while RF generates sd-envy-free allocation in 48% of preference profiles which is an improvement of 12% in RF. Moreover, given the nature of the Flow Algorithm, it is possible to improve this behavior by designing a more fair and easy-to-implement algorithm while keeping higher efficiency (interim) in this spirit.

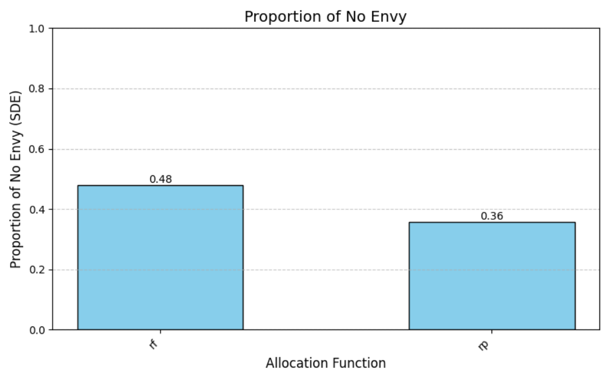


FIGURE 2. Ratio of No-envy for ABM, RP, PS, RF

**Observation: RF generates a more equalized allocation than RP and PS in about 60% of profiles.**

In comparing RP, PS, and RF, Table 13 displays the percentage of preference profiles where the row method outperforms the column method. RF generates a more equalized allocation than RP in about 61.5% of profiles and then PS in roughly 59.7% of profiles.

| | RP | PS | RF |
|---|---|---|---|
| RP | | 0.479 | 0.289 |
| PS | 0.502 | | 0.402 |
| RF | 0.615 | 0.597 | |

TABLE 13. Better Performance for RP, PS, and RF when comparing to RP, PS, and RF

Then we analyze to which extent the RF is more equalized. In Figure 3, we compare RF and PS across each preference profile. The x-axis represents each preference profile, while the y-axis represents the variance difference between PS and RF, specifically calculated as the variance of PS minus the variance of RF. A positive value on the graph signifies that the allocation under PS is less equalized than that under RF, and conversely for negative values. Moreover, in those 59.7% preference profiles, the variance under PS is much larger than the variance under RF with the maximum 0.047 while in the 40.2% preference profiles, the variance under RP is much larger than variance under PS with the maximum 0.01. So we can state that for the preference profile PS is much equalized, the difference between RF and PS is not large.

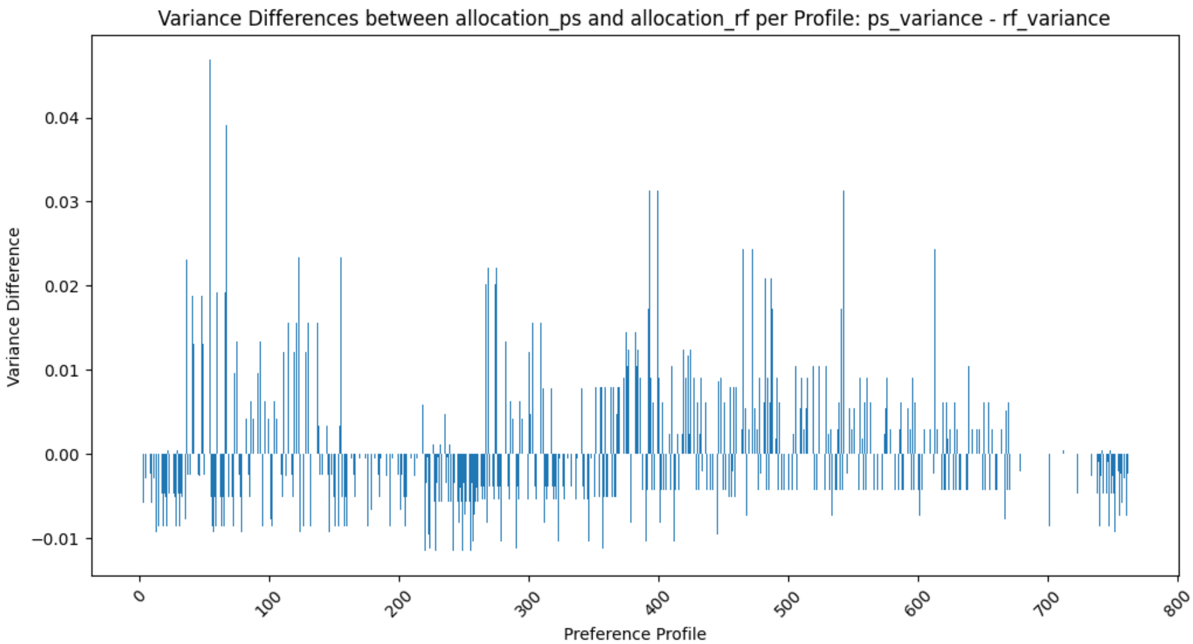


FIGURE 3. Difference between PS and RF

## 7. Conclusion

In conclusion, this paper deals with the random allocation problem when there is the initial order of objects, for example, allocating perishable goods to agents, or allocating students to tire structure schools, etc. When the agent's personal preference is inconsistent with the initial order, we provide the criteria (stronger than ordinal efficiency) that: an object, $a$, should not be given to a *'heavily-biased agent'* when the *'less-biased agent'* is still available. Moreover, the natural fairness criteria come out (stronger than Lexi-Envy-free): all such agents with the same 'bias' to the object should have an equal chance to the object.

It is worth noting that when every object is at the same position in the order, it is reduced to the property called *interim favoring support* and *equal support equal claim* which can characterize Adaptive Boston Mechanism. To notice, Chen, Harless, and Jiao (2023) replace the notion of 'availability' with 'rank in preference' to characterize the Boston Mechanism. Moreover, to characterize the property when every object is not at the same position in the order, we propose the new simple method, FRD (Flow algorithm with rank dominance).

Lastly, we consider constructing a more efficient algorithm that is still easy to play and easy to understand by agents. Simply, we uniformly randomize the Flow algorithm, called Random Flow (RF in short). And it satisfies a new notion of efficiency, *'Interim Efficient'*. *'Interim Efficient'* requires the random allocation can decompose into a convex combination of probabilistic ordinally efficient allocations. The *Interim Efficiency* is logically squeezed between ordinal efficiency and ex-post Pareto Efficiency: every ordinally efficient allocation is *'Interim Efficient'* and every *'Interim Efficient'* allocation is Ex-post Pareto efficient (the converse is not true). We show Random Flow is ea interim efficient while Random Priority is not. However, RF is not strategy-proof, but weakly strategy-proof.

Besides the axiomatic analysis, we provide an experimental analysis of existing mechanisms. We observed RF generates no-envy in more preference profiles than RP with an improvement of 12%. From these observations, we can design and apply an easy-to-play algorithm following this spirit that is superior to RP with efficiency and fairness.

## References


Abdulkadiroğlu, Atila, and Tayfun Sönmez. 1998. "Random serial dictatorship and the core from random endowments in house allocation problems." *Econometrica* 66 (3): 689–701.

Abdulkadiroğlu, Atila, and Tayfun Sönmez. 2003a. "Ordinal efficiency and dominated sets of assignments." *Journal of Economic Theory* 112 (1): 157–172.

Abdulkadiroğlu, Atila, and Tayfun Sönmez. 2003b. "School choice: A mechanism design approach." *American economic review* 93 (3): 729–747.

Aydin Alptekinoglu, Gerdus Benadè. 2023. "Achieving Rawlsian Justice in Food Rescue."

Aziz, Haris, Simon Mackenzie, Lirong Xia, and Chun Ye. 2015. "Ex post Efficiency of Random Assignments.." In *AAMAS,*: 1639–1640.

Bogomolnaia, Anna. 2015. "Random assignment: Redefining the serial rule." *Journal of Economic Theory* 158: 308–318.

Bogomolnaia, Anna, and Hervé Moulin. 2001. "A new solution to the random assignment problem." *Journal of Economic theory* 100 (2): 295–328.

Chen, Yajing, Patrick Harless, and Zhenhua Jiao. 2023. "The fractional Boston random assignment rule and its axiomatic characterization." *Review of Economic Design*: 1–23.

Gale, David, and Lloyd S Shapley. 1962. "College admissions and the stability of marriage." *The American Mathematical Monthly* 69 (1): 9–15.
Han, Xiang. 2023. "A theory of fair random allocation under priorities." *Theoretical Economics*.
Harless, Patrick. 2019. "Efficient rules for probabilistic assignment." *Journal of Mathematical Economics* 84: 107–116.
Kesten, Onur, and M Utku Ünver. 2015. "A theory of school-choice lotteries." *Theoretical Economics* 10 (2): 543–595.
Kojima, Fuhito, and M Utku Ünver. 2014. "The "Boston" school-choice mechanism: an axiomatic approach." *Economic Theory* 55: 515–544.
Liu, Peng, and Huaxia Zeng. 2019. "Random assignments on preference domains with a tier structure." *Journal of Mathematical Economics* 84: 176–194.
Mennle, Timo, and Sven Seuken. 2014. "The Naıve versus the Adaptive Boston Mechanism." *arXiv preprint arXiv:1406.3327*.
Mennle, Timo, and Sven Seuken. 2021. "Partial strategyproofness: Relaxing strategyproofness for the random assignment problem." *Journal of Economic Theory* 191: 105144.
Patrick, Harless. 2018. "Immediate Acceptance and Weak Priorities: An Adaptation that Preserves Efficiency and Respects Rank." *Working Paper*.
Prendergast, Canice. 2022. "The allocation of food to food banks." *Journal of Political Economy* 130 (8): 1993–2017.
Ramezanian, Rasoul, and Mehdi Feizi. 2021. "Stepwise ordinal efficiency for the random assignment problem." *Journal of Mathematical Economics* 92: 60–65.
Ramezanian, Rasoul, and Mehdi Feizi. 2022. "Robust ex-post Pareto efficiency and fairness in random assignments: Two impossibility results." *Games and Economic Behavior* 135: 356–367.
Richter, Michael, and Ariel Rubinstein. 2024. "Unilateral stability in matching problems." *Journal of Economic Theory* 216: 105780.
Roth, Alvin E, Uriel G Rothblum, and John H Vande Vate. 1993. "Stable matchings, optimal assignments, and linear programming." *Mathematics of operations research* 18 (4): 803–828.
Salgado-Torres, Alfredo. 2013. "Incomplete information and costly signaling in college admissions."Technical report, Working paper.
Xie, Jiarui. 2024. "Games under the Tiered Deferred Acceptance Mechanism." *arXiv preprint arXiv:2406.00455*.

# Appendix A. Proof for Relation between New axioms and Classic Axioms

## A.1. Proof for Proposition 1

PROOF. We show if the allocation is not ordinally efficient, then it violates Property 1. Fix preference profile $R_N$ and assume the allocation $P$ is not ordinally efficient, then there exists a (probabilistic) improvement circle $\tau$ that $a_k \tau_{i_k} a_{k+1}$ if and only if $a_k R_{i_k} a_{k+1}$

and $P_{i_k a_{k+1}} > 0$.

$$a_1 \tau_{i_1} a_2 \ldots a_n \tau_{i_n} a_{n+1}$$

We denote $a_{n+1} = a_1$. Now we define $Z(i_K; a_{K+1}; P) = \max_{k=1}^{n}\{Z(i_k; a_{k+1}; P)\}$. We have $Z(i_K; a_{K+1}; P) \geq Z(i_{K+1}; a_{K+2}; P) > Z(i_{K+1}; a_{K+1}; P)$. However, $\sum_{x \in \bar{U}(R_{i_{K+1}}, a_{K+1})} P_{i_{k+1} x} \leq 1 - P_{i_{K+1} a_{K+2}} < 1$ which violate Property 1. □

### A.2. Proof for Proposition 2

Proof. Fix preference profile $R_N$ and assume the allocation $P$ is not *lexi-envy-free*, then there exist agent $i$ with the preference $a_1 R_i a_2 \ldots R_i a_n$ and agent $j$ and the number $K \in \{1, \ldots, n\}$ such that $P_{ia_K} < P_{j a_K}$, while $P_{ia_k} = P_{j a_k}$ for all $k < K$. If $Z(i; a_K; P) = Z(j : a_K; P)$, we have $\sum_{x \in \bar{U}(R_i, a_K)} P_{ix} < \sum_{x \in U(R_j, a_K)} P_{jx} + P_{j a_K} \leq 1$, which violate the Property 2.

□

### A.3. Proof for Proposition 4

Proof. (a) Consider the preference profile:

$$1, 2 : abcd; 3, 4 : bacd$$

And the allocation:

$$1, 2 : (\frac{1}{2}, 0, \frac{1}{3}, \frac{1}{6})$$

$$3, 4 : (0, \frac{1}{2}, \frac{1}{6}, \frac{1}{3})$$

The allocation does not violate lexi-envy-free and Property 1, but violates Property 2 because $P_{1c} > P_{3c} > 0$ but agent 3 does not satisfy with the $\bar{U}(R_3; c)$.

(b) Consider the preference profile:

$$1, 2 : abcd; 3, 4 : bacd$$

And the allocation:

$$1, 2 : (\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 0)$$

$$3, 4 : (0, \frac{1}{4}, \frac{1}{4}, \frac{1}{2})$$

The allocation does not violate ordinally efficient, Property 2, but violates Property 1 because $P_{1b} > 0$ but agent 3,4 do not satisfy with $b$.

□

### A.4. Proof for Proposition ??

PROOF. We show if the allocation is not ordinally efficient, then it violates Property 1. Fix preference profile $R_N$ and assume the allocation $P$ is not ordinally efficient, then there exists a (probabilistic) improvement circle $\tau$ that $a_k \tau_{i_k} a_{k+1}$ if and only if $a_k R_{i_k} a_{k+1}$ and $P_{i_k a_{k+1}} > 0$.

$$a_1 \tau_{i_1} a_2 \ldots a_n \tau_{i_n} a_{n+1}$$

We denote $a_{n+1} = a_1$. Now we define $Z(i_K; a_{K+1}; P) = \max_{k=1}^{n}\{Z(i_k; a_{k+1}; P)\}$. We have $Z(i_K; a_{K+1}; P) \geq Z(i_{K+1}; a_{K+2}; P) > Z(i_{K+1}; a_{K+1}; P)$. However, $\sum_{x \in \bar{U}(R_{i_{K+1}}, a_{K+1})} P_{i_{k+1} x} \leq 1 - P_{i_{K+1} a_{K+2}} < 1$ which violate Property 1. □

## Appendix B. Characterization of Adaptive Boston Mechanism: Proposition 3

Whenever $R_N$, $a \in A$ and $P \in \mathcal{P}$, we define $N_a = \{i \in N : P_{ia} > 0\}$ be the set of agents who receive $a$ and $N_a^{wish} = \{i \in N : P_{ia} = 0 \text{ and } \exists x, aR_i x, \text{ s.t. } P_{ix} > 0\}$ be the set of agents who wish to receive $a$ to by replacing with other objects if it is possible.

PROPOSITION A1. *If an allocation satisfies Property 1, then for all $R_N$, all $a$,*

(a) $\max_{i \in N_a} Z(i; a; P) \leq \min_{i \in N_a^{wish}} Z(i; a; P)$.

(b) *For all $i$, $P_{ia} = 0$ and* $\sum_{x \in U(R_i; a)} P_{ix} < 1 \Rightarrow \sum_{j : Z(j; a; P) \leq Z(i; a; P)} P_{ja} = 1$.

PROOF.

LEMMA A1. *Condition (a) and Condition (b) are equivalent.*

⇒Fix $R_N$ and $a$, and assume $P$ satisfies condition (a). Pick $i \in N_a^{wish}$, then we have $\sum_{j : Z(j; a; P) \leq Z(i; a; P)} P_{ja} = \sum_{j \in N_a} P_{ja} + 0 = 1$.

⇐ Pick $i^* = \arg\min_{i \in N_a^{wish}} Z(i; a; P)$, then we have $\sum_{j \in N_a} P_{ja} = 1$. Condition (b) implies $N_a \subseteq \{j : Z(j; a; P) \leq Z(i^*; a; P)\}$ and $\max_{i \in N_a} Z(i; a; P) \leq \min_{i \in N_a^{wish}} Z(i; a; P)$.

LEMMA A2. *For all $R_N$, all $a$, all $i$, if an allocation satisfies Property 1, then* $\max_{i\in N_a} Z(i;a;P) \leq \min_{i\in N_a^{wish}} Z(i;a;P)$.

Fix $R_N$ and $P$, we have $N_a^{wish}$ and $N_a$. Pick $i^* = \arg\max_{i\in N_a} Z(i;a;P)$. We show $Z(i;a;P) \geq Z(i^*;a;P)$ for all $i \in N_a^{wish}$. Suppose not, then exist $i \in N_a^{wish}$ that $Z(i;a;P) < Z(i^*;a;P)$, then by Property 1, we have $\sum_{x\in \bar{U}(R_i;a)} P_{ix} = 1$ which contradict to the fact that $i \in N_a^{wish}$. Then it implies $\max_{i\in N_a} Z(i;a;P) \leq \min_{i\in N_a^{wish}} Z(i;a;P)$.

□

PROOF. Now, given $R_N$ we denote $P^*$ as the assignment of Adaptive Boston Mechanism and denote $P$ as the assignment satisfies Property 2 and Property 1. Now we show, for all $i$, all $a$ we have $P_{ia} = P^*_{ia}$. At first, we show for all $i$ and $\forall a$ that $Z(i;a;P) = 1$, then $P_{ia} = P^*_{ia}$.

We suppose there exists agent $i$ that $P_{ia} < P^*_{ia}$. There are two cases: $P_{ia} > 0$ or $P_{ia} = 0$.

**Claim 1.1:** For all $i$ and $\forall a$ that $Z(i;a;P) = 1$ and $P_{ia} > 0$, then $P_{ia} = P^*_{ia}$.

Property 2 implies $P_{ia} = P_{ja} > 0$ for all $j$ that $Z(j;a;P) = 1$, then it also implies $r(j;a) = 1$. The Property 2 implies $P_{ia} = P_{ja} < P^*_{ia} = P^*_{ja}, \forall i, j$ that $r(i;a) = r(j;a) = 1$, which means $\sum_{i:(r(i;a)=1} P_{ia} < \sum_{i:(r(i;a)=1} P^*_{ia} \leq 1$. Then Property 1 implies $\forall i$ that $r(i;a) = 1$, $P_{ia} = 1$, which is a contradiction. Then we conclude $P_{ia} = P^*_{ia}$. The case $P_{ia} > P^*_{ia}$ is symmetry.

**Claim 1.2:** For all $i$ and $\forall a$ that $Z(i;a;P) = 1$ and $P_{ia} = 0$, then $P_{ia} = P^*_{ia}$.

If $P_{ia} = 0$, Property 1 implies either $\sum_{k:Z(k;a;P)\leq 1} P_{ka} = 1$ or $\sum_{x\in U(R_i;a)} P_{ix} = 1$. The latter and Claim 1.1 implies $P^*_{ia} = P_{ia} = 0$. Now if $\sum_{x\in U(R_i;a)} P_{ix} < 1$, then $a$ is fully allocated to agents who rank $a$ in the top, namely for all $j$ who $r(j;a) = 1$. By Claim 1.1, we know $P_{ja} = P^*_{ja}$ for all $j$, $r(j;a) = 1$. Then it implies $P_{ia} = 1 - \sum_{k:Z(k;a;P)\leq 1} P_{ka} = 0 = P^*_{ia}$

We conclude for all $i$ and $\forall a$ that $Z(i;a;P) = 1$, then $P_{ia} = P^*_{ia}$.

Now assume for all $i$ and all $a$ that $Z(i;a;P) = z$, we have $P_{ia} = P^*_{ia}$ for $z = 1, .., k-1$, we show for all $i$ and all $a$ that $Z(i;a;P) = k$, then $P_{ia} = P^*_{ia}$. By contradiction, suppose, there exists agent $i$ such that $P_{ia} < P^*_{ia}$. Again, there are two cases $P_{ia} > 0$ or $P_{ia} = 0$.

**Claim 2.1:** For all $i$ and $\forall a$ that $Z(i;a;P) = k$ and $P_{ia} > 0$, then $P_{ia} = P^*_{ia}$.

There are two cases:

(a) If $P_{ia} = P_{ja}$ for all $j$ that $Z(j;a;P) = k$. Then we have $P_{ia} = P_{ja} < P^*_{ia} = P^*_{ja}$ for all

$j$ that $Z(j;a;P) = k$. It implies $\sum_{i:Z(i;a;P)\leq k} P_{ia} < \sum_{i:Z(i;a;P)\leq k} P^*_{ia} \leq 1$. Then Property 1 implies $\sum_{x\in\overline{U}(R_i;a)} P_{ix} = 1$ for all $i$ that $Z(j;a;P) \leq k$. However, we have

$$\sum_{x\in\overline{U}(R_i;a)} P_{ix} = \sum_{x\in U(R_i;a)} P_{ix} + P_{ia} = \sum_{x\in U(R_i;a)} P^*_{ix} + P_{ia} < \sum_{x\in\overline{U}(R_i;a)} P^*_{ix} \leq 1$$

Which is a contradiction. Then $P^*_{ia} = P^*_{ia}$.

(b) If there exists $j$ that $P_{ia} > P_{ja}$. Property 2 implies

$$\sum_{x\in U(R_j;a)} P_{jx} + P_{ja} = \sum_{x:Z(j;x;P)<k} P^*_{jx} + P_{ja} = 1.$$

Now suppose $P^*_{ja} < P_{ja}$ (it can not be $P^*_{ja} > P_{ja}$), then we have $P^*_{ia} > P_{ia} > P_{ja} > P^*_{ja}$. By Property 2, we have $\sum_{x\in U(R_j;a)} P^*_{jx} + P^*_{ja} = 1$, then $P^*_{ja} = P_{ja}$, a contradiction. Therefore, we have $P^*_{ja} = P_{ja}$. Consider agent $i$, we have

$$\sum_{i:Z(i;a;P)\leq k} P_{ia} < \sum_{i:Z(i;a;P)\leq k} P^*_{ia} \leq 1 \text{ and } \sum_{x\in\overline{U}(R_i;a)} P_{ix} < \sum_{x\in\overline{U}(R_i;a)} P^*_{ix} \leq 1$$

which is a contradiction. Then we have $P_{ia} = P^*_{ia}$.

Then we conclude $P_{ia} = P^*_{ia}$ for all $i$ and $\forall a$ that $Z(i;a;P) = k$ and $P_{ia} > 0$.
**Claim 2.2:** For all $i$ and $\forall a$ that $Z(i;a;P) = k$ and $P_{ia} = 0$, then $P_{ia} = P^*_{ia}$.
By Property 1, we have either $\sum_{x\in U(R_i;a)} P_{ix} = 1$ or $\sum_{j:Z(j;a;P)\leq Z(i;a;P)} P_{ja} = 1$.

- $\sum_{x\in U(R_i;a)} P_{ix} = 1$ implies $\sum_{x\in U(R_i;a)} P^*_{ix} = \sum_{x\in U(R_i;a)} P_{ix} = 1$, then $P^*_{ia} = P_{ia} = 0$.
- $\sum_{j:Z(j;a;P)\leq Z(i;a;P)} P_{ja} = 1$ and Claim 2.1 states for all $i$ that $Z(i;a;P) = k$ and $0 < P_{ia}$ we have $P_{ia} = P^*_{ia}$. Therefore

$$\sum_{j:Z(j;a;P)\leq Z(i;a;P)} P^*_{ja} = \sum_{j:Z(j;a;P)\leq Z(i;a;P)} P_{ja} = 1. \Rightarrow P^*_{ia} = 0$$

Then we conclude $P_{ia} = P^*_{ia}$ for all $i$ and $\forall a$. We complete the proof. For necessity, it exactly follows the definition.

□

# Appendix C. Characterization of FRD: Proposition 6

PROOF. Whenever $\succ$, $P$, $a \in A$, $i, j \in N$, and $P_{ia} > 0$ we define $i \overset{a}{\Longrightarrow} j$ if either $i \overset{a}{\longrightarrow} j$ or $i \sim j$.

Whenever $R_N$, $a \in A$ and $P \in \mathcal{P}$, we define $N_a = \{i \in N : P_{ia} > 0\}$ be the set of agents who receive $a$ and $N_a^{wish} = \{i \in N : P_{ia} = 0 \text{ and } \exists x, aR_ix, \text{ s.t. } P_{ix} > 0\}$ be the set of agents who wish to receive $a$ to by replacing with other objects if it is possible.

LEMMA A3. *If Property 1 holds, then* $[P_{ia} = 0$ *and* $\sum_{x \in U(R_i;a)} P_{ix} < 1] \Rightarrow \sum_{j : j \overset{a}{\Longrightarrow} i} P_{ja} = 1.$

PROOF. We show $j \overset{a}{\Longrightarrow} i$ for all $i \in N_a^{wish}$ and for all $j \in N_a$, which implies $a$ is full distributed among agents $j \overset{a}{\Longrightarrow} i$ whenever $i \in N_a^{wish}$, thus complete the proof. Pick $j^* \in N_a$ with the highest $Z(i;a;P)$ among agents who have he highest $\pi_{\overline{*}(a)}$.

Suppose not, there exist $i \in N_a^{wish}$ that $i \overset{a}{\longrightarrow} j^*$. By Property 1, we have $\sum_{x \in \bar{U}(R_i;a)} P_{ix} = 1$ which violate the fact that $i \in N_a^{wish}$. □

To prove the theorem, we need a few notations.

Given $\overline{*}(i;A)$, $\overline{*}(i;a)$ is called the last peak for agent $i$ when for all $b \in \overline{L}(i;\overline{*}(i;a))$ that $\overline{*}(i;b) = \overline{*}(i;a)$. $\overline{L}(R_i;a)$ is the weak lower counter set of $a$ for agent $i$ under $R_i$. If $\overline{*}(i;a)$ is not the last peak we denote $\underline{*}(i;a)$ as the peak immediately after $a$, namely there exists $b$ with smallest $r(i;b)$ that $\overline{*}(i;b) \neq \overline{*}(i;a)$. Then we define $*(i;a)$ as the set containing $a$ as following:

(a) $*(i;a) = \{b : r(i;\overline{*}(i;a)) \leq r(i;b) < r(i;\underline{*}(i;a))\}$ if $\overline{*}(i;a)$ is not last peak.

(b) $*(i;a) = \{b : r(i;\overline{*}(i;a)) \leq r(i;b) \leq |A|\}$ if $\overline{*}(i;a)$ is last peak.

Whenever $R_N$, we define $\pi^1 = \min\{\pi_x : x \in \bigcup_{i \in N} \overline{*}(i;A)\}$ as the rank of minimal peak, and $\pi^k = \min\{\pi_x : x \in \bigcup_{i \in N} \overline{*}(i;A), \pi_x > \pi^{k-1}\}$ as the rank of $k-th$ peak. Then, given $R_N$, there is the largest rank $\pi^K$. For each $\pi^k$, we define $I(\pi^k) = \{i : \exists b, \overline{*}(i;b) \in \pi(\pi^k)\}$ be the set of agents who have one peak $b \in \pi(\pi^k)$. Then for all $i \in I(\pi^k)$, we denote the peak as $a_i^k$.

Now, given $R_N$ we denote $P^\pi = F(R_N)$ and denote $P$ as the assignment satisfies Property* 2 and Property* 1 that $P \neq P^\pi$.

**Claim 1:** For all $i \in I(\pi^1)$, $\forall a \in *(i;a_i^1)$ we have $P_{ia} = P^\pi_{ia}$.

This is easily obtained from the Proposition 3. People who are familiar with Proof of Proposition 3 can jump to Claim 2.

**Claim 1.1:** For all $i \in I(\pi^1)$ and $\forall a$ that $\hat{Z}(i;a;p) = 1$, then $P_{ia} = P^{\pi}_{ia}$.

We suppose there exists agent $i$ that $P_{ia} < P^{\pi}_{ia}$. There are two cases: $P_{ia} > 0$ or $P_{ia} = 0$.

**Claim 1.1.1:** For all $i \in I(\pi^1)$ and $\forall a$ that $\hat{Z}(i;a;p) = 1$ and $P_{ia} > 0$, then $P_{ia} = P^{\pi}_{ia}$.

Property* 2 implies if $P_{ia} = P_{ja} > 0$ for all $j$ that $\hat{Z}(j;a;p) = 1$, then it also implies $r(j;a) = 1$. The Property* 2 implies $P_{ia} = P_{ja} < P^{\pi}_{ia} = P^{\pi}_{ja}$, $\forall i, j$ that $r(i;a) = r(j;a) = 1$, which means $\sum_{i:(r(i;a)=1} P_{ia} < \sum_{i:(r(i;a)=1} P^{\pi}_{ia} \leq 1$. Then Property* 1 implies $\forall i$ that $r(i;a) = 1$, $P_{ia} = 1$, which is a contradiction. Then we conclude $P_{ia} = P^{\pi}_{ia}$. The case $P_{ia} > P^{\pi}_{ia}$ is symmetry.

**Claim 1.1.2:** For all $i \in I(\pi^1)$ and $\forall a$ that $\hat{Z}(i;a;p) = 1$ and $P_{ia} = 0$, then $P_{ia} = P^{\pi}_{ia}$.

By Property* 1, if $P_{ia} = 0$, then either $\sum_{x \in U(R_i;a)} P_{ix} = 1$ or $\sum_{j:j \overset{a}{\Longrightarrow} i} P_{ja} = 1$. The former one implies $P^{*}_{ia} = 0 = P_{ix}$ immediately. If $\sum_{x \in U(R_i;a)} P_{ix} < 1$, then $a$ must be fully allocated to agents who rank $a$ in the top, namely for all $j$, $r(j;a) = 1$. By Claim 1.1, we know $P_{ja} = P^{\pi}_{ja}$ for all $j$, $r(j;a) = 1$. Then it implies $P^{\pi}_{ia} = P_{ia} = 0$.

Now assume for all For all $i \in I(\pi^1)$ and $\forall a$ that $\hat{Z}(i;a;p) = z$, we have $P_{ia} = P^{\pi}_{ia}$ for $z = 1, .., k-1$, we show for all $i \in I(\pi^1)$ and $\forall a$ that $\hat{Z}(i;a;p) = k$, then $P_{ia} = P^{\pi}_{ia}$.

**Claim 1.2:** For all $i \in I(\pi^1)$ and $\forall a$ that $\hat{Z}(i;a;p) = k$, then $P_{ia} = P^{\pi}_{ia}$.

**Claim 1.2.1:** For all $i \in I(\pi^1)$ and $\forall a$ that $\hat{Z}(i;a;p) = k$ and $P_{ia} > 0$, then $P_{ia} = P^{\pi}_{ia}$.

By contradiction, suppose, $P_{ia} < P^{\pi}_{ia}$. Again, $P_{ia} > P^{\pi}_{ia}$ is symmetric. There are two cases:

(a) If $P_{ia} = P_{ja}$ for all $j$ that $\hat{Z}(j;a;p) = k$. Then we have $P_{ia} = P_{ja} < P^{\pi}_{ia} = P^{\pi}_{ja}$ for all $j$ that $\hat{Z}(j;a;p) = k$. It implies $\sum_{i:\hat{Z}(i;a;p) \leq k} P_{ia} < \sum_{i:\hat{Z}(i;a;p) \leq k} P^{\pi}_{ia}$ and $\sum_{x \in \overline{U}(R_i;a)} P_{ix} = 1$ for all $i$ that $\hat{Z}(j;a;p) \leq k$. However, we have

$$\sum_{x \in \overline{U}(R_i;a)} P_{ix} = \sum_{x \in U(R_i;a)} P_{ix} + P_{ia} = \sum_{x \in U(R_i;a)} P^{\pi}_{ix} + P_{ia} < \sum_{x \in \overline{U}(R_i;a)} P^{\pi}_{ix} \leq 1$$

Which is a contradiction. Then $P^{\pi}_{ia} = P^{\pi}_{ia}$ for all all $i$ that $\hat{Z}(j;a;p) = k$ if $P_{ia} = P_{ja}$ for all $j$ that $\hat{Z}(j;a;p) = k$.

(b) If there exists $j$ that $P_{ia} > P_{ja}$. Property* 2 implies $\sum_{x \in U(R_j;a)} P_{jx} + P_{ja} = \sum_{x:Z(j;x;P)<k} P^{\pi}_{jx} +$

$P_{ja} = 1$. Now suppose $P^{\pi}_{ja} < P_{ja}$ (it can not be $P^{\pi}_{ja} > P_{ja}$), then we have $P^{\pi}_{ia} > P_{ia} > P_{ja} > P^{\pi}_{ja}$. By Property* 2, we have $\sum_{x\in U(R_j;a)} P^{\pi}_{jx} + P^{\pi}_{ja} = 1$, then $P^{\pi}_{ja} = P_{ja}$, a contradiction. Therefore, we have $P^{\pi}_{ja} = P_{ja}$. Now for $i$, we have $\sum_{i:\hat{Z}(i;a;p)\leq k} P_{ia} < \sum_{i:\hat{Z}(i;a;p)\leq k} P^{\pi}_{ia} \leq 1$ and $\sum_{x\in \overline{U}(R_i;a)} P_{ix} < \sum_{x\in \overline{U}(R_i;a)} P^{\pi}_{ix} \leq 1$ , which is a contradiction. Then we have $P_{ia} = P^{\pi}_{ia}$.

Then we conclude $P_{ia} = P^{\pi}_{ia}$ for all $i \in I(\pi^1)$ and $\forall a$ that $\hat{Z}(i;a;p) = k$ and $P_{ia} > 0$.

**Claim 1.2.2:** For all $i \in I(\pi^1)$ and $\forall a$ that $\hat{Z}(i;a;p) = k$ and $P_{ia} = 0$, then $P_{ia} = P^{\pi}_{ia}$.

By Property* 1, if $\sum_{x\in U(R_i;a)} < 1$ (otherwise, $P^{*}_{ia} = P_{ia} = 0$), we have $\sum_{j:j\overset{a}{\Longrightarrow}i} P_{ja} = 1$ and

Claim 1.2.1 states $0 < P_{ia} = P^{\pi}_{ia}$ for all $i$ that $\hat{Z}(i;a;p) = k$, also $P_{ia} = P^{\pi}_{ia}$ for all $i$ that $\hat{Z}(i;a;p) = j$ for all $j = 1,\dots,k-1$. Therefore $\sum_{j:j\overset{a}{\Longrightarrow}i} P^{\pi}_{ja} = \sum_{j:j\overset{a}{\Longrightarrow}i} P_{ja} = 1$. It implies $P^{\pi}_{ia} = 0$.

Then we conclude $P_{ia} = P^{\pi}_{ia}$ for all $i \in I(\pi^1)$ and $\forall a \in *(i;a^1_i)$.

**Claim 2:** Assume for all $i \in I(\pi^z)$, $\forall a \in *(i;a^z_i)$ we have $P_{ia} = P^{\pi}_{ia}$ for all $z = 1,\dots,k-1$, then we show $i \in I(\pi^k)$, $\forall a \in *(i;a^k_i)$ we have $P_{ia} = P^{\pi}_{ia}$.

**Claim 2.1:** For all $i \in I(\pi^k)$ and $\forall a$ that $\hat{Z}(i;a;p) = 1$, then $P_{ia} = P^{\pi}_{ia}$.

We suppose there exists agent $i$ that $P_{ia} < P^{\pi}_{ia}$. There are two cases: $P_{ia} > 0$ or $P_{ia} = 0$.

**Claim 2.1.1:** For all $i \in I(\pi^k)$ and $\forall a$ that $\hat{Z}(i;a;p) = 1$ and $P_{ia} > 0$, then $P_{ia} = P^{\pi}_{ia}$.

There are two cases:

(a) If $P_{ia} = P_{ja}$ for all $j$ that $\hat{Z}(j;a;p) = 1$. Then we have $P_{ia} = P_{ja} < P^{\pi}_{ia} = P^{\pi}_{ja}$ for all $j$ that $\hat{Z}(j;a;p) = 1$. It implies $\sum_i P_{ia} < \sum_i P^{\pi}_{ia} \leq 1$ and $\sum_{x\in \overline{U}(R_i;a)} P_{ix} = 1$ for all $i$ that $\hat{Z}(i;a;p) = 1$. However, we have

$$\sum_{x\in \overline{U}(R_i;a)} P_{ix} = \sum_{x\in \bigcup_{z<k} *(i;a^z_i)} P_{ix} + P_{ia} = \sum_{x\in \bigcup_{z<k} *(i;a^z_i)} P^{\pi}_{ix} + P_{ia} < \sum_{x\in \overline{U}(R_i;a)} P^{\pi}_{ix} \leq 1$$

Which is a contradiction. Then $P^{\pi}_{ia} = P^{\pi}_{ia}$ for all all $i$ that $\hat{Z}(j;a;p) = 1$ if $P_{ia} = P_{ja}$ for all $j$ that $\hat{Z}(j;a;p) = 1$.

(b) If there exists $j$ that $P_{ia} > P_{ja}$. Property* 2 implies

$$\sum_{x \in U(R_j;a)} P_{jx} + P_{ja} = \sum_{x \in \underset{z<k}{\cup} *(j;a_j^z)} P_{jx} + P_{ja} = \sum_{x \in \underset{z<k}{\cup} *(j;a_j^z)} P^{\pi}_{jx} + P_{ja} = 1$$

Now suppose $P^{\pi}_{ja} < P_{ja}$ (it can not be $P^{\pi}_{ja} > P_{ja}$), then we have $P^{\pi}_{ia} > P_{ia} > P_{ja} > P^{\pi}_{ja}$. By Property* 2, we have $\sum\limits_{x \in U(R_j;a)} P^{\pi}_{jx} + P^{\pi}_{ja} = 1$, then $P^{\pi}_{ja} = P_{ja}$, a contradiction.

Therefore, we have $P^{\pi}_{ja} = P_{ja}$.

Now for $i$, we have

$$\sum_{i \in \underset{z<k}{\cup} I(\pi^z)} P_{ia} + \sum_{i \in I(\pi^k), \hat{Z}(i;a;p)=1} P_{ia} = \sum_{i \in \underset{z<k}{\cup} I(\pi^z)} P^{\pi}_{ia} + \sum_{i \in I(\pi^k), \hat{Z}(i;a;p)=1} P_{ia} < 1$$

which is a contradiction. Then we have $P_{ia} = P^{\pi}_{ia}$.

Then we conclude $P_{ia} = P^{\pi}_{ia}$ for all $i \in I(\pi^k)$ and $\forall a$ that $\hat{Z}(i;a;p) = 1$ and $P_{ia} > 0$.

**Claim 2.1.2:** For all $i \in I(\pi^k)$ and $\forall a$ that $\hat{Z}(i;a;p) = 1$ and $P_{ia} = 0$, then $P_{ia} = P^{\pi}_{ia}$.

By Property* 1, if $\sum\limits_{x \in U(R_i;a)} < 1$ (otherwise, $P^{*}_{ia} = P_{ia} = 0$), we have $\sum\limits_{j:j \overset{a}{\Longrightarrow} i} P_{ja} = 1$ and

Claim 2.1.1 states $0 < P_{ia} = P^{\pi}_{ia}$ for all $i$ that $i \in I(\pi^k)$ and $\hat{Z}(i;a;p) = 1$, also $P_{ia} = P^{\pi}_{ia}$ for all $i$ that $i \in I(\pi^z)$ for all $z = 1, \ldots, k-1$. Therefore $\sum\limits_{j:j \overset{a}{\Longrightarrow} i} P^{\pi}_{ja} = \sum\limits_{j:j \overset{a}{\Longrightarrow} i} P_{ja} = 1$. It implies $P^{\pi}_{ia} = 0$.

Now assume for all For all $i \in I(\pi^k)$ and $\forall a$ that $\hat{Z}(i;a;p) = z$, then $P_{ia} = P^{\pi}_{ia}$ for $z = 1, .., m-1$, we show for all $i \in I(\pi^k)$ and $\forall a$ that $\hat{Z}(i;a;p) = m$, then $P_{ia} = P^{\pi}_{ia}$.

**Claim 2.2:** For all $i \in I(\pi^k)$ and $\forall a$ that $\hat{Z}(i;a;p) = m$, then $P_{ia} = P^{\pi}_{ia}$.

**Claim 2.2.1:** For all $i \in I(\pi^k)$ and $\forall a$ that $\hat{Z}(i;a;p) = m$ and $P_{ia} > 0$, then $P_{ia} = P^{\pi}_{ia}$.

By contradiction, suppose, $P_{ia} < P^{\pi}_{ia}$. Then there are two cases:

(a) If $P_{ia} = P_{ja}$ for all $j$ that $\hat{Z}(j;a;p) = m$. Then we have $P_{ia} = P_{ja} < P^{\pi}_{ia} = P^{\pi}_{ja}$ for all $j$ that $\hat{Z}(j;a;p) = m$. It implies $\sum\limits_{i} P_{ia} < \sum\limits_{i} P^{\pi}_{ia} \leq 1$ and $\sum\limits_{x \in \overline{U}(R_i;a)} P_{ix} = 1$ for all $i$ that $\hat{Z}(j;a;p) = m$. However, we have

$$\sum_{x\in\overline{U}(R_i;a)} P_{ix} = \sum_{x\in\bigcup_{z<k}*(i;a_i^z)} P_{ix} + \sum_{x\in*(i;a_i^k),Z(i;x;P)<m} P_{ix} + P_{ia}$$
$$= \sum_{x\in\bigcup_{z<k}*(i;a_i^z)} P_{ix}^{\pi} + \sum_{x\in*(i;a_i^k),Z(i;x;P)<m} P_{ix}^{\pi} + P_{ia}$$
$$< \sum_{x\in\overline{U}(R_i;a)} P_{ix}^{\pi} \leq 1$$

Which is a contradiction. Then $P_{ia}^{\pi} = P_{ia}^{\pi}$ for all all $i$ that $\hat{Z}(j;a;p) = m$ if $P_{ia} = P_{ja}$ for all $j$ that $\hat{Z}(j;a;p) = m$.

(b) If there exists $j$ that $P_{ia} > P_{ja}$. Property* 2 implies

$$\sum_{x\in\overline{U}(R_j;a)} P_{jx} = \sum_{x\in\bigcup_{z<k}*(j;a_j^z)} P_{jx} + \sum_{x\in*(j;a_i^k),Z(j;x;P)<m} P_{jx} + P_{ja}$$
$$= \sum_{x\in\bigcup_{z<k}*(j;a_j^z)} P_{jx}^{\pi} + \sum_{x\in*(j;a_j^k),Z(j;x;P)<m} P_{jx}^{\pi} + P_{ja} = 1$$

Now suppose $P_{ja}^{\pi} < P_{ja}$, then we have $P_{ia}^{\pi} > P_{ia} > P_{ja} > P_{ja}^{\pi}$. By Property* 2, we have $\sum_{x\in\bigcup_{z<k}*(j;a_j^z)} P_{jx}^{\pi} + \sum_{x\in*(j;a_j^k),Z(j;x;P)<m} P_{jx}^{\pi} + P_{ja}^{\pi} = 1$, then $P_{ja}^{\pi} = P_{ja}$, a contradiction. Therefore, we have $P_{ja}^{\pi} = P_{ja}$.

Now for $i$, we have

$$\sum_{i\in\bigcup_{z<k}I(\pi^z)} P_{ia} + \sum_{i\in I(\pi^k),Z(i;a;P)<m} P_{ia} + \sum_{i\in I(\pi^k),\hat{Z}(i;a;p)=m} P_{ia}$$
$$= \sum_{i\in\bigcup_{z<k}I(\pi^z)} P_{ia}^{\pi} + \sum_{i\in I(\pi^k),\hat{Z}(i;a;p)<m} P_{ia}^{\pi} + \sum_{i\in I(\pi^k),\hat{Z}(i;a;p)=m} P_{ia} < 1$$

which is a contradiction. Then we have $P_{ia} = P_{ia}^{\pi}$.

Then we conclude $P_{ia} = P_{ia}^{\pi}$ for all $i \in I(\pi^k)$ and $\forall a$ that $\hat{Z}(i;a;p) = m$ and $P_{ia} > 0$.

**Claim 2.2.2:** For all $i \in I(\pi^k)$ and $\forall a$ that $\hat{Z}(i;a;p) = m$ and $P_{ia} = 0$, then $P_{ia} = P_{ia}^{\pi}$.

By Property* 1, if $\sum_{x \in U(R_i;a)} < 1$ (otherwise, $P^*_{ia} = P_{ia} = 0$), we have $\sum_{j: j \overset{a}{\Longrightarrow} i} P_{ja} = 1$ and Claim 2.2.1 states $P_{ia} = P^{\pi}_{ia}$ for all $i \in I(\pi^k)$ that $\hat{Z}(i;a;p) < m$, and $0 < P_{ia} = P^{\pi}_{ia}$ for all $i \in I(\pi^k)$ that $\hat{Z}(i;a;p) = m$, and for all $i \in I(\pi^z)$ for all $z = 1, \ldots, k-1$. Therefore $\sum_{j: j \overset{a}{\Longrightarrow} i} P^{\pi}_{ja} = \sum_{j: j \overset{a}{\Longrightarrow} i} P_{ja} = 1$. It implies $P^{\pi}_{ia} = 0$.

Then we conclude $P_{ia} = P^{\pi}_{ia}$ for all $i \in I(\pi^k)$ and $\forall a \in *(i; a^k_i)$. We complete the proof. For necessity, it exactly follows the definition of $F^{\pi}$.

□

## Appendix D. Proof for Proposition 7

Proof. (a) Consider Example 1:

'Interim Efficient' requires a random assignment that can be decomposed into a convex combination of ordinally efficient random assignments, then it is ex-post Pareto efficient. Now we show the converse is not true.

Example 1. *Assume there are 4 agents and 4 objects, and the preference profile is the following:*

(A1)

*1: acbd*
*2: adbc*
*3: bcad*
*4: bdac*

*Consider a random assignment*

| | a | b | c | d |
|---|---|---|---|---|
| 1 | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 0 |
| 2 | $\frac{1}{2}$ | 0 | 0 | $\frac{1}{2}$ |
| 3 | 0 | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ |
| 4 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 |

*There are only two deterministic assignments, and it is easy to check those two are efficient because they could be represented by a priority order (1 2 3 4) and (2 1 4 3):*

| | a | b | c | d |
|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 | 1 |
| 3 | 0 | 1 | 0 | 0 |
| 4 | 0 | 0 | 1 | 0 |

| | a | b | c | d |
|---|---|---|---|---|
| 1 | 0 | 0 | 1 | 0 |
| 2 | 1 | 0 | 0 | 0 |
| 3 | 0 | 0 | 0 | 1 |
| 4 | 0 | 1 | 0 | 0 |

TABLE A1. Two deterministic assignments

*Also, this random assignment can't be decomposed into the convex combination of probabilistic ordinally efficient assignments because all '0' should be unchanged after convex combination, then agent 3 will trade object 'd' with agent 4 for object 'c', therefore this random assignment is ex-post Pareto efficient, not 'Interim Efficient'.*

(b) Consider Example 2:

EXAMPLE 2. *Assume there are 4 agents and 4 objects, and the preference profile is the following:*

(A2)
*1,2: badc*
*3,4: abcd*

*Consider a random assignment*

| | a | b | c | d |
|---|---|---|---|---|
| 1 | $\frac{1}{8}$ | $\frac{3}{8}$ | $\frac{1}{8}$ | $\frac{3}{8}$ |
| 2 | $\frac{1}{8}$ | $\frac{3}{8}$ | $\frac{1}{8}$ | $\frac{3}{8}$ |
| 3 | $\frac{3}{8}$ | $\frac{1}{8}$ | $\frac{3}{8}$ | $\frac{1}{8}$ |
| 4 | $\frac{3}{8}$ | $\frac{1}{8}$ | $\frac{3}{8}$ | $\frac{1}{8}$ |

*It is the average of ordinally efficient random assignments of:*

| | a | b | c | d |
|---|---|---|---|---|
| 1 | $\frac{1}{4}$ | $\frac{1}{2}$ | 0 | $\frac{1}{4}$ |
| 2 | $\frac{1}{4}$ | $\frac{1}{2}$ | 0 | $\frac{1}{4}$ |
| 3 | $\frac{1}{4}$ | 0 | $\frac{1}{2}$ | $\frac{1}{4}$ |
| 4 | $\frac{1}{4}$ | 0 | $\frac{1}{2}$ | $\frac{1}{4}$ |

| | a | b | c | d |
|---|---|---|---|---|
| 1 | 0 | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{2}$ |
| 2 | 0 | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{2}$ |
| 3 | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | 0 |
| 4 | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | 0 |

TABLE A2. Two Random assignments

*But this random assignment is not ordinally efficient.*

| 1: | *adbc* |
|---|---|
| 2: | *acbd* |
| 3: | *abdc* |
| 4: | *abcd* |

TABLE A3. $R_N$

| | a | b | c | d |
|---|---|---|---|---|
| 1 | $\frac{1}{4}$ | 0 | $\frac{1}{24}$ | $\frac{17}{24}$ |
| 2 | $\frac{1}{4}$ | 0 | $\frac{17}{24}$ | $\frac{1}{24}$ |
| 3 | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{1}{12}$ | $\frac{1}{6}$ |
| 4 | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{1}{6}$ | $\frac{1}{12}$ |

TABLE A4. Allocation of *RP*

TABLE A5. Violation of 'Interim Efficient': RP

(c) Consider the preference profile:

In this preference profile, if $P$ is ordinally efficient, then

(i) If $P_{1c} > 0$, then $P_{2d} = P_{4d} = 0$.

(ii) If $P_{4d} > 0$, then $P_{1c} = P_{3c} = 0$.

| | a | b | c | d |
|---|---|---|---|---|
| 1 | | 0 | + | |
| 2 | | 0 | | 0 |
| 3 | | | | |
| 4 | | | | 0 |

| | a | b | c | d |
|---|---|---|---|---|
| 1 | | 0 | 0 | |
| 2 | | 0 | | |
| 3 | | | 0 | |
| 4 | | | | + |

TABLE A6. Two possibility

Then we try to construct the support of random assignment in Table A5 such that the Table A6 must be satisfied. In the end, we will have a contradiction.

**Claim 1:** Now, suppose $P_{1c} > 0$, then it must be Table A7 with the weight $\frac{1}{24}$:

| | a | b | c | d |
|---|---|---|---|---|
| 1 | 0 | 0 | 1 | 0 |
| 2 | 1 | 0 | 0 | 0 |
| 3 | 0 | 0 | 0 | 1 |
| 4 | 0 | 1 | 0 | 0 |

TABLE A7. Possible deterministic assignments when $P_{1c} > 0$

**Claim 2:** Suppose $P_{4d} > 0$, then it must be Table A8 with weight $\frac{1}{12}$.

| | a | b | c | d |
|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 |
| 2 | 0 | 0 | 1 | 0 |
| 3 | 0 | 1 | 0 | 0 |
| 4 | 0 | 0 | 0 | 1 |

TABLE A8. Possible deterministic assignments when $P_{4d} > 0$

Now, we have:

| | a | b | c | d |
|---|---|---|---|---|
| 1 | $\frac{1}{12}$ | 0 | $\frac{1}{24}$ | 0 |
| 2 | $\frac{1}{24}$ | 0 | $\frac{1}{12}$ | 0 |
| 3 | 0 | $\frac{1}{12}$ | 0 | $\frac{1}{24}$ |
| 4 | 0 | $\frac{1}{24}$ | 0 | $\frac{1}{12}$ |

TABLE A9. $\frac{1}{12}$ Table A8+$\frac{1}{24}$ Table A7

Given Table A5 and Table A9, we know the support of Table A5 must contains following deterministic assignments:

(i) When $P_{1a} = 1$:

| | a | b | c | d |
|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 | 1 |
| 3 | 0 | 1 | 0 | 0 |
| 4 | 0 | 0 | 1 | 0 |

| | a | b | c | d |
|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 | 1 |
| 3 | 0 | 0 | 1 | 0 |
| 4 | 0 | 1 | 0 | 0 |

TABLE A10. Total Weight $\frac{1}{24}$

and

| | a | b | c | d |
|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 |
| 2 | 0 | 0 | 1 | 0 |
| 3 | 0 | 0 | 0 | 1 |
| 4 | 0 | 1 | 0 | 0 |

TABLE A11. With Wight $\frac{1}{8}$

(ii) When $P_{1d} = 1$:

| | a | b | c | d |
|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 1 |
| 2 | 1 | 0 | 0 | 0 |
| 3 | 0 | 0 | 1 | 0 |
| 4 | 0 | 1 | 0 | 0 |

| | a | b | c | d |
|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 1 |
| 2 | 1 | 0 | 0 | 0 |
| 3 | 0 | 1 | 0 | 0 |
| 4 | 0 | 0 | 1 | 0 |

TABLE A12. Total Weight $\frac{5}{24}$

and

| | a | b | c | d |
|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 1 |
| 2 | 0 | 0 | 1 | 0 |
| 3 | 0 | 1 | 0 | 0 |
| 4 | 1 | 0 | 0 | 0 |

| | a | b | c | d |
|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 1 |
| 2 | 0 | 0 | 1 | 0 |
| 3 | 1 | 0 | 0 | 0 |
| 4 | 0 | 1 | 0 | 0 |

TABLE A13. Wight with $\frac{1}{4}$ each

Now because $P_{4a} = \frac{1}{4}$ in Table A5, then we know the third deterministic assignment of Table A13 must weight $\frac{1}{4}$. And $P_{3d} = \frac{1}{6}$ implies Table A11 must be weight $\frac{1}{8} = \frac{1}{6} - \frac{1}{24}$. Then we have

| | a | b | c | d |
|---|---|---|---|---|
| 1 | $\frac{5}{24}$ | 0 | $\frac{1}{24}$ | $\frac{1}{2}$ |
| 2 | $\frac{1}{24}$ | 0 | $\frac{17}{24}$ | 0 |
| 3 | $\frac{1}{4}$ | $\frac{7}{24}$ | 0 | $\frac{1}{6}$ |
| 4 | $\frac{1}{4}$ | $\frac{10}{24}$ | 0 | $\frac{1}{12}$ |

TABLE A14. $\frac{1}{12}$ Table A8+$\frac{1}{24}$ Table A7+ $\frac{1}{4}$ Table A13 each+$\frac{1}{8}$ Table A11

Given the current assignment, we have the lottery left $\frac{1}{4}$ and four deterministic assignments:

| | a | b | c | d |
|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 | 1 |
| 3 | 0 | 1 | 0 | 0 |
| 4 | 0 | 0 | 1 | 0 |

| | a | b | c | d |
|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 | 1 |
| 3 | 0 | 0 | 1 | 0 |
| 4 | 0 | 1 | 0 | 0 |

| | a | b | c | d |
|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 1 |
| 2 | 1 | 0 | 0 | 0 |
| 3 | 0 | 0 | 1 | 0 |
| 4 | 0 | 1 | 0 | 0 |

| | a | b | c | d |
|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 1 |
| 2 | 1 | 0 | 0 | 0 |
| 3 | 0 | 1 | 0 | 0 |
| 4 | 0 | 0 | 1 | 0 |

TABLE A15. Total weight $\frac{1}{4}$

Now denote the weight for each deterministic assignments in Table A15 from left to right as $x$, $y$, $z$, $w$. Then we have the following equations:

$$\begin{cases} x + w = \frac{1}{6} = P_{4c} \\ x + w = \frac{5}{24} = \frac{1}{2} - \frac{7}{24} = P_{3b} - \frac{7}{24} \end{cases}$$

Which is a contradiction.

(d) We know $F^{\pi}$ is ordinally efficient for every $\pi \in \Pi$, then $RF$ is interim efficiency,.

□

## Appendix E. Proof for Proposition 8

PROOF. interim efficiency is proved, now we show weakly envy-freeness and weakly strategy-proofness.

**Claim 1:** RF is weakly Envy-free.

LEMMA A4. *For all $R_N$, all $\pi \in \Pi$, all i that $R_i = a_1 R_i a_2 R_i \dots R_i a_n$, if there is an adjacent preference $R'$ with and $a_{K+1} R' a_K$ for some K, then $P_{ia}(R) \leq P_{ia}(R_i', R_{-i})$.*

PROOF. Denote $p_{ia_{K+1}}$ be the probability to get $a_{K+1}$ under preference $R_i$ and $p'_{ia_{K+1}}$ be the probability to get $a_{K+1}$ under preference $R'$. we have

$$R_i = a_1 R_i \dots a_K R_i a_{K+1} R_i a_n \tag{A3}$$

$$R_i' = a_1 R_i' \dots a_{K+1} R_i' a_K R_i' a_n \tag{A4}$$

At first, we have $p_{ia_k} = p'_{ia_k}, \forall k \leq K$. By procedure of $F^{\pi}$ there exist $K^*$ such that $\pi^{-1}(K^*) = a_K$ and $i \in N(a_K, R)$ in short $N(a_K)$, in other words, i would be allocated $a_K$ in round $K^*$.

If $\pi^{-1}(K^*+1) = a_{K+1}$, then $p_{ia_{K+1}} = \phi(a_{K+1}, P_{K^*+1}, R_{K^*+1})$ while $p'_{ia_{K+1}} = \phi(a_{K+1}, P_{K^{**}+1}, R_{K^{**}+1})$ for $K^{**} > K^*$ if $\sum_i p'_{ia_{K+1}} < 1$, or 0. In latter case, $p_{ia_{K+1}} \geq p'_{ia_{K+1}} = 0$ is obvious, In first case, we have $\sum_{x \in \bar{U}(R_i, a_K)} p_{ix} \leq \sum_{x \in \bar{U}(R', a_K)} p'_{ix} \Leftrightarrow 1 - \sum_{x \in \bar{U}(R_i, a_K)} p_{ix} \geq 1 - \sum_{x \in \bar{U}(R', a_K)} p'_{ix}$ which means agent $i$ with $R'$ has less or equal availability in round $K^{**}+1$ compared to round $K^*+1$ with $R_i$. Then $1 - \sum_{x \in \bar{U}(R_i, a_K)} p_{ix} \geq \phi(a_{K+1}, P_{K^{**}+1}, R_{K^{**}+1})$. Then we need to show $\phi(a_{K+1}, P_{K^*+1}, R_{K^*+1}) \geq 1 - \sum_{x \in \bar{U}(R_i, a_K)} p_{ix}$. If $\frac{1 - \sum_{i \in N} p_{ia}}{|N(a_{K+1})|} \geq 1 - \sum_{x \in \bar{U}(R_i, a_{K+1})} p_{ix}$, it's true because $\phi(a_{K+1}, P_{K^*+1}, R_{K^*+1}) = 1 - \sum_{x \in \bar{U}(R_i, a_K)} p_{ix}$.

On the other hand, if $\frac{1 - \sum_{i \in N} p_{ia}}{|N(a_{K+1})|} < 1 - \sum_{x \in \bar{U}(R_i, a_{K+1})} p_{ix}$, then equal share is the most agent $i$ can get to object $a_{K+1}$, therefore $p_{ia_{K+1}} \geq p'_{ia_{K+1}}$ too.

If for some positive k, $\pi^{-1}(K^* + k) = a_{K+1}$, the analysis for agent $i$ with the preference $R_i$ will not change because he will not be allocated any object until round $K^* + k$. On the other hand, the analysis for agent $i$ with the preference $R'$ will be different. Moreover, agent $i$ can't have a higher probability to get object $a_{K+1}$ because $1 - \sum_{x \in \bar{U}(R', a_K)} p'_{ix}$ is even lower as there is a non-negative probability to be allocated before round $K^* + k$.

Otherwise, $a_{K+1}$ is consumed before $K^*$ round, then $a_{K+1}$ either be allocated fully before $K^*$ round, or it goes back to the end of $\pi$, thus, there will exist some positive k, $\pi^{-1}(K^* + k) = a_{K+1}$, and same analysis above.

Now, only one possibility remains: $a_{K+1}$ be allocated fully before $K^*$ round, then $p_{ia_{K+1}} = p'_{ia_{K+1}} = 0$, finish proof. □

LEMMA A5. *RF is weakly envy-free.*

PROOF. Fix a preference profile $\succsim$ with $\succsim_i = a_1 \succsim_i a_2 \succsim \cdots \succsim_i a_n$, then there exist agent $j$ such that $RF_i(\succsim) \succsim_i^{sd} RF_j(\succsim)$. We need to show $RF_i(\succsim) = RF_j(\succsim)$. Let's start from the

$a_1$. Assume $\succsim_j(1) = a_k \neq a_1$, then there is $\frac{1}{2}$ probability that $a_1$ proceeds $a_k$ and $\frac{1}{2}$ probability that $a_k$ proceeds $a_1$. Then $p_{ia_1} = \frac{1}{2}(\frac{1}{N(a_1)} + \frac{1}{N_2(a_1)}) > p_{ja_1} = \frac{1}{2}(0 + \frac{1}{N_2(a_1)})$ which is a contraction to $RF_i(\succsim) \succsim_i^{sd} RF_j(\succsim)$, therefore $\succsim_j(1) = a_1 \Rightarrow p_{ia_1} = p_{ja_1}$.

Then let's look at $a_2$. Given the above analysis, w.o.l.g., let's assume $\succsim_j(2) = a_k$:

$$\succsim_i = a_1 \succsim_i a_2 \succsim_i \cdots \succsim_i a_k \succsim_i \ldots \tag{A5}$$

$$\succsim_j = a_1 \succsim_j a_k \succsim_j \cdots \succsim_i a_2 \succsim_i \ldots \tag{A6}$$

Let's start to look at $a_2$ and $a_k$, there are also two possibilities, either $a_k$ proceeds $a_2$ or $a_2$ proceeds $a_k$.

If $a_k$ proceeds $a_2$, then when $a_2$ appears, we have $1 - \sum_x p_{jx} < 1 - \sum_x p_{ix}$, then (average) $p_{ja_2} > p_{ia_2}$ because either they have the same probability (could be 0) to access object $a_2$ where the equal share is lower than $1 - \sum_x p_{jx}$ under some $\pi$ or agent $i$ has higher probability to access object $a_2$ when the equal share is higher than $1 - \sum_x p_{ix}$ under some $\pi$, therefore, the average of probability to access $a_2$ will be higher for agent $i$.

Now if $a_2$ proceeds $a_k$, there are two possibilities, when $p_{ia_2}$ is 0, then $p_{ja_2} = p_{ia_2}$ and when $p_{ia_2} \neq 0$, then $p_{ia_2} \geq p_{ja_2}$. , therefore, the average of probability to access $a_2$ will at least equal for agent $i$ compared to agent $j$. To sum up (by average), we get $p_{ia_2} > p_{ja_2}$ which is a contradiction. Therefore $p_{ja_2} = p_{ia_2}$.

It's enough to make induction for any $a_k$ given $p_{ia_{k-1}} = p_{ja_{k-1}}$, by assuming $\succsim_j(k) \neq \succsim_i(k)$, then we get either $p^{\pi}_{ia_k} = p^{\pi}_{ja_k}$ (could be 0) or $p^{\pi}_{ia_k} > p^{\pi}_{ja_k}$, and by averaging, $p_{ia_k} > p_{ja_k}$, which is a contradiction. Then $\succsim_j(k) = \succsim_i(k) \Rightarrow p_{ia_k} = p_{jak}$ for all $a_k$. □

**Claim 2:** RF is weakly strategy-proof.

Now we prove weakly strategy-proofness. We will show RF is upper invariance and Swap monotonicity (Mennle and Seuken (2021)).

Definition A1. *Adjacent preference*

*Given $R_i$, we say $R_j$ is an adjacent preference of $R_i$ if there exist $K \in [n]$, such that*

(a) $o(i;K+1) = o(j;K)$ *and* $o(j;K+1) = o(i;K)$

(b) $o(i;k) = o(j;k)$ *for all* $k \in [n] \setminus \{K, K+1\}$

Then given a preference $R$, we denote the set of adjacent preferences as $\delta(R)$.

Axiom 1. *(Swap monotonicity)*

*A mechanism f is swap monotonic if, for all agents $i \in N$, all R and all $R'_i \in \delta(R)$ with $aR_ib$ but $bR'_ia$, one of the following holds:*

(a) *either: $f_i(R'_i, R_{-i}) = f_i(R)$,*

(b) *or: $f_{i,b}(R'_i, R_{-i}) > f_{i,b}(R)$.*

In other words, swap monotonicity requires that the mechanism reacts to the swap in a direct and monotonic way: If the swap that brings b forward affects the agent's assignment at all, then at least its assignment for b must be affected directly. Moreover, this change must be monotonic in the sense that the agent's assignment for b must increase when b is reportedly more preferred.

Axiom 2. *(Upper invariance)*

*A mechanism g is upper invariant if, for all agents $i \in N$, all R and all $R'_i \in \delta(R)$ with $aR_ib$ but $bR'_ia$, we have $f_{i,x}(R) = f_{i,x}(R'_i, R_{-i})$ for all $x \in U(R_i, a)$*

Axiom 3. *(Lower invariance)*

*A mechanism g is lower invariant if, for all agents $i \in N$, all R and all $R'_i \in \delta(R)$ with $aR_ib$ but $bR'_ia$, we have $f_{i,x}(R) = f_{i,x}(R'_i, R_{-i})$ for all $x \in L(R_i, b)$*

From Mennel and Seuken Mennle and Seuken (2021), strategy-proofness can be decomposed into three axioms, Swap monotonicity, upper invariance, and lower invariance. If one mechanism only satisfies the swap monotonic and upper invariant then it is weakly strategy-proof.

**Claim 2.1: RF satisfies upper invariance.**

To notice $F^\pi$ will allocate agent object $a$ to $i$ in round $k$ only when every object that is better than a is allocated (according to agent **i** preference) and $\pi(k) = a$ for some k, then $R'_i(j) = R'_i(j)$ for all $j < k$ implies $p_{ia_j} = p'_{ia_j}$ for all $j < k$ which means that RF is upper invariance because it's an average over $\pi$.

**Claim 2.2: RF satisfies Swap monotonicity.**

Swap monotonicity $\Rightarrow$ Fix a preference profile $R$ with $R_i : a_1R_ia_2R_i \ldots$. Consider a preference $R'_i$ such that $a_l R'_i a_k$ with $l > k$. if we take $R'_i$ as a truth preference, We've that $\forall\pi$ if $F^\pi_i(R)R^{sd}_iF^\pi_i(R'_i, R_{-i})$,then $F^\pi_i(R) = F^\pi_i(R'_i, R_{-i})$. Now, we consider if they are not comparable by stochastic dominance. Consider $a_l$, there are two possibilities that either $a_l$ proceeds $a_k$ or $a_k$ proceeds $a_l$. Then we can repeat the analysis in Lemma A4 to check there $p'^\pi_{ia_l} \geq p^\pi_{ia_l}$ for all $\pi$, and there exist a $\pi$ such that it's strict. Then $p'_{ia_l} > p_{ia_l}$ after averaging. Immediate result from Mennle and Seuken (2021). RF is weakly strategy-proof.

□