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Lindahl meets Condorcet?

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Abstract

Although a Condorcet winner commands a majority in its favor, there is no guarantee of unanimity. In a Lindahl equilibrium, a suitably chosen system of personalized transfers and prices ensures unanimity, but there is no guarantee of a majority vote in its favor. Do Lindahl equilibria decentralize Condorcet winners? In a setting where voters' preferences are satiated, characterized by bliss points, this paper proposes a new balancedness condition which is satisfied when a Condorcet winner lies within the interior of the convex hull of voters' bliss points. We show that such a political compromise between the most preferred policies of different voter types can be decentralized as Lindahl equilibria.

Keywords: Bliss points; Condorcet winner; Lindahl equilibria, balancedness.
JEL classification: D50, D61, D71

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"The law of majority voting itself rests on an agreement and implies that there has been on at least one occasion unanimity." - J-J Rousseau, *The Social Contract*, Rousseau (1968).

1 Introduction

Condorcet winners,[1] when they exist, are focal points in determining the outcome of public choice (see, for instance, Black (1958); Sen (2020)). However, a minority of voters may disagree with the majority's decision. One way to solve this problem is to examine whether the decision is supported by Lindahl taxes or subsidies (Lindahl (1919)), calculated using personalized prices that account for the marginal trade-offs implied by voters' preferences. If so, the policy with majority support would also be unanimous. Conversely, a Lindahl equilibrium that is also a Condorcet winner would have the desirable feature of being implementable via a majority vote (see Gul and Pesendorfer (2020)).

In an ideal world, Lindahl should meet Condorcet, but could they?

Following Black (1958), the extensive literature on public choice and Condorcet winners assumes that voter preferences are characterized by bliss points over public policy or social states.[2] Examples include single-peaked preferences in one-dimensional settings or Euclidean preferences in multidimensional settings. Condorcet winners have been shown to exist when voters have such preferences (see, for example, Black (1958); Plott (1967); Caplin and Nalebuff (1988, 1991); Pivato (2015); Maskin (2023)).[3] Preferences over public policy, when characterized by bliss points, allow voters to hold different conceptions of the "ideal" public policy.

A key characteristic of preferences with bliss points is that they display satiation. However, the existence of a Lindahl equilibrium is generally established in settings where satiation is explicitly ruled out (see, for instance, Foley (1970), Roberts (1973), Mas-Colell (1980), Gul and Pesendorfer (2020), Carvajal and Song (2022)).

[1] In the remainder of the paper, unless explicitly stated otherwise, we always refer to a Condorcet winner under a non-unanimity voting rule.

[2] The terms "public policy" and "social states" will be used interchangeably throughout the paper.

[3] Maskin (2023) works with ideological preferences that are single-peaked.

Ghosal and Polemarchakis (1999) examined the issue of the existence of Lindahl equilibria with satiated preferences. They show that for a fixed distribution of revenue, a Lindahl equilibrium may not exist. Moreover, beyond the standard assumptions, a sufficient condition to decentralize a Pareto optimal state as a Lindahl equilibrium with transfers is that the Pareto state be irreducible. This means that for any partition of the set of individuals into two non-empty groups, each individual in one group can be made strictly better off in another feasible state.

Our first result shows that when a Pareto optimal state is irreducible, it can never be a Condorcet winner. At first glance, this result could be interpreted to imply that the outcomes supported as Condorcet winners cannot be decentralized via Lindahl equilibria. However, since irreducibility is a sufficient but not a necessary condition for the existence of a Lindahl equilibrium, it remains possible for a Lindahl equilibrium to exist even without this condition being satisfied.

To study settings where irreducibility is never satisfied but Condorcet winners still exist, we introduce a new condition – *balancedness* – that ensures the existence of a Lindahl equilibrium with transfers.[4] In particular, we demonstrate the existence of a type-symmetric Lindahl equilibrium with transfers.

Balancedness is sometimes, but not always, consistent with the existence of a Condorcet winner. Specifically, there are settings where irreducibility fails and no Condorcet winners exist, yet balancedness is satisfied, ensuring the existence of a Lindahl equilibrium. Thus, markets with personalized prices and adjusted revenue distributions can ensure unanimity for a public policy even if it is not a Condorcet winner. Although the existence of a Condorcet winner[5] does not always guarantee balancedness, we provide a condition under which it does: namely, when the Condorcet winner lies in the interior of the convex hull of voters' bliss points. This means that the policy being decentralized is a political compromise between the most preferred policies of different voter types. When

[4]It may be of interest to note that both irreducibility and balancedness are non-individualistic assumptions, as they are conditions on the entire configuration of preferences across all agents.

[5]The existence of a Condorcet winner under an m-majority rule requires that $1-m$ is the size of the largest coalition opposing a policy chosen by it, a point applicable to both finite (as in this paper) and infinite voter settings (as in Caplin and Nalebuff (1988); Caplin and Nalebuff (1991)).

this is not the case, we provide an example showing that a Condorcet winner cannot be decentralized.

Furthermore, the pairwise balancedness condition, used to prove the existence of a Condorcet winner in Plott (1967), implies but is not implied by the balancedness condition studied here. Our results can also be interpreted in relation to a version of the first and second welfare theorems for the Lindahl equilibria. Finally, we demonstrate how our analysis can be extended to the provision of "social states" or "public policies" with quasi-linear preferences and non-zero marginal costs of production.

The remainder of the paper is organized as follows. The next section introduces the model and some preliminary results, followed by a section focused on the use of the balancedness condition in settings with Condorcet winners. The last section concludes with a brief discussion of applications/interpretations of the balancedness condition.

2 The Model and Initial Results

2.1 Social States and Condorcet Winners

We consider a finite set N of agents/voters (where n denotes the cadinality of the set) with preferences over a set of feasible public policies/social states $Q \subset \mathbb{R}^L$. The preferences of an agent are represented by a continuous utility function $u_i : Q \to \mathbb{R}$. We work with the notion of a type-symmetric voting (and subsequently, type-symmetric Lindahl equilibria) where agents with identical preferences over social states vote in the same way (and face the same personalized prices and transfers and choose the same allocation).[6] Let T be a partition over N, each element of which is denoted by a type t such that whenever $i, j \in t$, $u_i = u_j$. The set of voters (society) can be summarized by a distribution of types μ in T with μ_t denoting the fraction of the set of agents of type t. Clearly, $\mu(T) = 1$. A society is $S = (Q, T, \{u_i, \mu_t : i \in T\})$.

[6]This notion allows also for a direct extention and comparison of our results to those obtained for models with a continuum of agents studied in the seminal papers on m-majority voting e.g. Caplin and Nalebuff (1988, 1991). See also Sen (2020) for an exposition of type-symmetric voting with a finite number of voters.

An individual of type t is satiated at $q \in Q$ if and only if $u_t(q) \geq u_t(q'), \forall q' \in Q$. An individual of type t is locally satiated at $q \in Q$ if there exists a neighborhood of q, $v_q \subset Q$, such that $u_t(q) \geq u_t(q'), \forall q' \in v_q$.

Given q, for a collection of types $T' \subseteq T$, $u^{T'}(q) = \{u_t(q) : i \in T'\}$.

A feasible q is Pareto dominated by a feasible q' if $u^T(q') > u^T(q)$ i.e. $u_t(q') \geq u_t(q)$ for all $t \in T$ with strich inequality for at least one $t \in T$. A feasible q is Pareto optimal if it is not Pareto dominated by any feasible q'. Note that in the above setting, preferences are satiated. The existence of a Pareto optimal q with satiated preferences is not evident: it is established by Lemma 1 in Ghosal and Polemarchakis (1999) with compactness of Q.

A feasible q is dominated by a feasible q' for a collection of agents $T' \subseteq T$, if $u^{T'}(q') > u^{T'}(q)$. A feasible social state q is dominated by a feasible social state q' under a m-majority voting rule, if there is a collection of agents $T' \subseteq T$ such that $u^{T'}(q') > u^{T'}(q)$ and $\mu(T') > m$ where $\mu(T') = \sum_{t \in T'} \mu_t, T' \subseteq T$.

We are now in a position to state the definition of a Condorcet winner:

Definition 1. *A feasible social state $q \in Q$ is a m-majority Condorcet winner if it is not dominated by any feasible social state under a m-majority voting rule.*

2.2 The Economy and Lindahl equilibria

The economy consists of a set of agents N and a firm. Different commodities, objects of exchange, correspond to different dimensions of social states for each type; this is necessary for the variables that affect the preferences of each individual of type t to be the objects of choice of that individual. The technology of the firm will be specified so that the commodities chosen by different types of individuals correspond to a specific social state at a feasible allocation in the economy. The characteristics of individuals and of the firm reflect the set of social states and the preferences of individuals.

Commodities are indexed by $(l, t), l = 1, .., L, t \in T$. Let $\mathbf{t}$ denote the cardinality of the set T. A bundle of commodities is $x = (..., x_{l,i}, ...)$. The domain of bundles of commodities is $\mathbb{R}^{L\mathbf{t}}$.

An individual of type t is described by the pair (X_t, u_t) where $X_t \subset \mathbb{R}^L$ is the exchange set for the individual, with $X_t = Q$ and $u_t : X_t \to \mathbb{R}$ is their ordinal utility function.

The firm is characterized by its technology, $Y \subset \mathbb{R}^{L\mathbf{t}}$ consisting of production bundles $y = (..., y_t, ...)$ where $y_t = y_0 \in Q$ for each $t \in T$.

The economy associated with a society S is $E = \{T, Y, (X_t, u_t, \mu_t : t \in T)\}$.

A state of the economy is an array (y, x), consisting of a production plan for the firm $y \in Y$, and for each individual an exchange bundle $x_t \in X_t$.

A state of the economy is feasible if and only if for all $t \in T$, $x_t = y_t$.

From the structure of the firm's technology and the exchange sets of individuals, an allocation is feasible if and only if there is a feasible social state $q \in Q$ such that $x_t = q$ and $y = (..., q, ...)$, $t \in T$.

There is an unambiguous association of a feasible social state with a feasible state of the economy, and vice versa.

The prices of commodities are $p = (..., p_t, ...)$. The value of a bundle of commodities, x at prices p is $(\mu. \times p) \cdot x = \sum_{t\in T} \mu_t p_t \cdot x_t$.

A revenue transfer is $w = (..., w_t, ...)$. At prices p and an allocation x, a transfer of revenue is w such that $\sum_{t\in T} \mu_t w_t = (\mu. \times p) \cdot x$.

A list $(y^*, (x_t^*, p_t^*)_{t\in T})$ is a type symmetric Lindhal equilibrium with transfers, if there exists a transfer of revenue $w^* = (w_t^*)_{t\in T}$ such that:

1. for each $t \in T$, x_t^* solves $\max_{x\in X_t}\ u_t(x)$ over $\{x \in Q : p_t^* \cdot x \leq w_t^*\}$,

2. y^* solves $\max_{y\in Y}\ (\mu. \times p^*) \cdot y$,

3. $x_t^* = y_t^*$, for each $t \in T$,

4. $\sum_{t\in T} \mu_t w_t^* = (\mu. \times p^*) \cdot y^*$.

The point is that in a type symmetric equilibrium prices and transfers at a symmetric Lindahl equilibrium are tailored to a type and not to each individual. From now on, we use the term Lindahl equilibrium and the type symmetric Lindahl equilibrium interchangeably. Since in any equilibrium there exists a $q^* \in Q$ such that $x_t^* = y_t^* = q^*$ for each

t, in what follows, we use a simpler notation $(q^*, (p_t^*)_{t\in T})$ to denote a Lindhal equilibrium with transfers.

Some comments.

First, condition 1 in the preceding definition implies that in a Lindahl equilibrium with transfers if $u_i(x_t) > u_t(x_t^*)$ for some $x_t \in X_t$, then $p_t^* \cdot x_t > w_t^*$. Suppose Condition 1 in the definition of a Lindahl equilibrium with transfers is replaced by the following auxillary condition: for each $t \in T$ if $u_t(x_t) \geq u_t(x_t^*)$ with $x_t \in X_t$, then $p_t^* \cdot x_t \geq w_t^*$; moreover, there exists at least one $t \in T$ such that if $u_t(x_t) > u_t(x_t^*)$ for some $x_t \in X_t$, then $p_t^* \cdot x_t > w_t^*$.

Then, we say that (x^*, y^*, p^*) is a Lindhal quasi-equilibrium with transfers w^*.

Second, if $y^* \in Y$ is in the interior of Y, then the existence of a Lindahl equilibrium requires $(\mu. \times p^*) \cdot y^* = 0$.

Third, for a fixed distribution of transfers, a Lindahl equilibrium need not exist, as demonstrated by the following example.

Example 1. *The set of feasible social states is $Q = [-1, 1]$. Individuals are indexed by $t \in T = \{1, 2\}$, $\mu_t = \frac{1}{2}, t \in T$ and have utility functions $u_1(q) = -(q-1)^2$ and $u_2(q) = -(q - \frac{1}{2})^2$ with domain over social states $X_t = Q$. The distribution of transfers is fixed with $w_t = 0$, $t \in T$. At prices $p = (p_1, p_2)$, the budget constraint for a type t individual is $p_t x_t \leq 0$. Suppose first that $p_1 + p_2 = 0$. If $p_1 = 0$, then $x_1 = 1$ while $x_2 = \frac{1}{2}$; if $p_1 > 0$, then $x_1 = 0$ while $x_2 = \frac{1}{2}$; if $p_1 < 0$, $x_1 = 1$ while $x_2 = 0$; all are contradictions. Alternatively, $p_1 + p_2 > 0$, from the maximization of the firm's profits, $y = (1, 1)$. If $x_1 = x_2 = 1$, then $p_1 + p_2 \leq 0$, a contradiction. By similar logic, $p_1 + p_2 < 0$ leads to a contradiction. Note, however, that there are transfer distributions for which Lindahl equilibria exist. Specifically if $w_1 + w_2 = 0$, $w_1 \in [\frac{1}{2}, 1]$ support $q \in [\frac{1}{2}, 1]$ as Lindahl equilibrium outcomes.*

2.3 Irreducibility, Condorcet Winners and Lindahl Equilibria

Proposition 1 in Ghosal and Polemarchakis (1999) established that the existence of a Lindahl equilibrium with transfers follows if the economy is irreducible. It will be convenient to recall the definition of irreducibility.

Definition 2. *A feasible social state $q \in Q$ is irreducible if and only if, for any partition of T into two non-empty sets T_1, T_2, there exists a feasible social state $q' \in Q$ such that for each $t \in T_1$ we have $u_t(q') > u_t(q)$.*

Let $Q^m \subset Q$ denote the set of $m - majority$ winners and let Q^{IR} denote the set of irreducible social states. The following proposition shows that these sets cannot have common elements under minimal assumptions.

Proposition 1. *Suppose that there exists a proper subset of T denoted by T' with $\mu(T') > m$. Then $Q^m \cap Q^{IR} = \emptyset$.*

Proof. The argument is based on contradiction. Suppose $Q^m \cap Q^{IR} \neq \emptyset$. Take $q \in Q^m \cap Q^{IR}$. By assumption, there exists a proper subset of T, say T' with $\mu(T') > m$. Take any such set. Since q is irreducible, there exists $q' \in Q$ such that every individual in T' strictly prefers q' to q. But since $\mu(T') > m$, q cannot be a m-majority winner. □

An implication of the preceding proposition is that when the social state satisfies irreducibility, it cannot be a Condorcet winner. On the face of it, this result would seem to imply that the set of Lindahl equilibria and the set of Condorcet winners are empty. However, irreducibility is a sufficient condition for the existence of a Lindahl equilibrium with transfers. So, Lindal equilibria with transfers may exist even without irreducibility being satisfied even when a Condorcert winner does not exist:

Example 2 (Motivating example)**.** *Consider 3 types of voters of equal measure, each with single-peaked preferences over 3 candidates/policies represented as points in $\mathbb{R}^2$. The bliss point of a voter of type 1 is $(-1, -2)$, a voter of type 2 is $(4, 0)$, and a voter of type 3 is $(-1, 4)$. The policies are given by $A = (0, 0)$, $B = (3, -2)$, and $C = (2, 3)$ as illustrated in Figure 1. Observe that there is a Condorcet cycle. In fact, none of the policies can be sustained. Suppose that voters start with a proposal B, then the types $\{1, 3\}$ prefer A, but then the types $\{3, 2\}$ prefer C, and then the types $\{2, 1\}$ prefer B. Neither of the three proposals is irreducible: in proposal A the subset of types $\{1, 3\}$ cannot be improved, in*

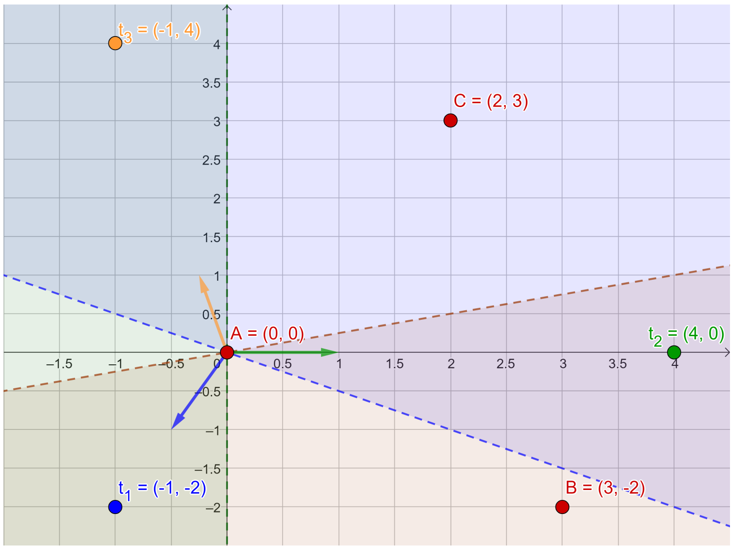

Figure 1: Example of the Condorcet cycle with 3 voters (with bliss points: t_1, t_2, t_3). None of the three policies (A, B, C) is irreducible, but A can be decentralized with personalized prices and transfers. Shaded areas denote votes' budget set (using color codes: blue for t_1, green for t_2 and orange for t_3) in the Lindhal equilibrium with tansfers.

C the subset of types $\{2, 3\}$ cannot be improved, while for B the subset of types $\{1, 2\}$ cannot be improved. However, proposal A can be decentralized in a Lindhal equilibrium, with personalized prices, e.g. $p_1 = (-8, -16)$, $p_2 = (12, 0)$ and $p_3 = (-4, 16)$ satisfying $p_1 + p_2 + p_3 = 0$ and transfers $w_1 = w_2 = w_3 = 0$.

There remains the question of when a Condorcet winner and a Lindahl equilibrium with transfers co-exist.

3 Bliss Points, Condorcet Winners and Symmetric Lindahl Equilibria

3.1 Single-Peaked preferences with bliss points over policies

To begin with, we study the case where $Q \subset \mathbb{R}$. The preferences of the agents over Q are anonymous and depend only on their type $t \in T \subseteq Q$, which denotes their bliss point with respect to policy q. Formally:

Assumption 1 (Single-peaked preferences)**.** *For each $t \in T$ a preference is represented by a utility $u_t : Q \to \mathbb{R}$ that is single-peaked, that is, it increases strictly in $\{q \in Q : q < t\}$ and decreases strictly in $\{q \in Q : q > t\}$.*

A typical example of such preferences is given by $u_t(q) = -(q-t)^2$ or a Euclidian norm.

We examine whether, for single-peaked preferences, a symmetric Lindhal equilibrium with transfers can decentralize a m-majority Condorcet winner. We start with the case where $m = \frac{1}{2}$.

Example 3 (Initial example)**.** *Consider a set of $N+1$ (N even) voters indexed by t from $T = \{0, \frac{1}{N}, \frac{2}{N}, \ldots, 1\}$ with $\mu_t = \frac{1}{N+1}$ and preferences over $q \in Q = [0,1] = T$ are given by $u_t(q) = -(q-t)^2$.*

By the median voter theorem, if agents vote on policies q_1, q_2 proposed by 2 candidates, each candidate will propose the same policy $q_1 = q_2 = 0.5$, that is, a bliss point for the median voter. Observe that $q = 0.5$ also solves the problem of maximizing the Social Welfare Function:

$$\max_{q \in [0,1]} -\frac{1}{N+1}\sum_{n=0}^{N}(q - \frac{n}{N})^2.$$

We now analyze the possibility of decentralizing $q = 0.5$ in the Lindhal equilibrium with transfers. Observe that $MU_{q,t} = -2q + 2t$ for $t \in T$ and therefore for personalized prices $p_t = 2t - 2q$ with fixed q each agent will choose the same policy. The sum of such prices is:

$$\frac{1}{N+1}\sum_{t \in T} p_t = \frac{2}{N+1}\sum_{n=0}^{N}(\frac{n}{N} - q) = \frac{2}{N+1}\left[\frac{N+1}{2} - (N+1)q\right] = 1 - 2q.$$

Observe that this equals zero for $q = 0.5$.

As a result of the aggregate price being equal to 0 (and only for this price), the firm's profit is zero for any q, the firm is therefore indifferent and therefore can supply $q = 0.5$. To finance the purchase of $q = 0.5$, we propose incomes $w_t = t - 0.5$. Observe that $\sum_{t \in T} \frac{1}{N+1} w_t = 0$, and hence such transfers are budget-balanced.

In the above example, we decentralize the median voter outcome (0.5) with personalized prices and transfers.

Some comments.

First, in fact, without appropriate transfers or profit allocation, a Lindhal equilibrium may not exist or decentralize the desired social outcome. In the following examples, we normalize the social outcome (to be decentralized) to 0 and, when doing so, set the transfer or profit allocation to $w_t = 0$ for each t. This is without loss of generality.

Second, the above example may suggest that the symmetry of the voters' (bliss points) distribution around the median voter is necessary for prices to sum up to zero. We will demonstrate that this is not the case. In fact, a Lindhal equilibrium can decentralize the median voter outcome for more general distributions of voters' bliss points.

Third, if preferences over social states are single-dimensional, we can, in fact, decentralize the social outcome (here the median voter) with two (non-zero) levels of prices only.

Fourth, the Lindhal equilibrium can be used to decentralize not only the median voter outcome but also other Pareto optimal allocations, in fact, any allocation that is contained in the interior of the convex hull of bliss points of the individuals/voters.

We illustrate all the above points in the following example.

Example 4 (Assymetric distributions and only 2 (non-zero) prices in Lindhal equilibrium)**.** *Consider the distribution of voters with preferences over* $q \in Q \ni 0$ *given by* $u_t(q) = -(q-t)^2$*. Assume that we want to decentralize the outcome* $q = 0$ *with a fraction* $\mu_- > 0$ *of the population with bliss points* t *below* 0*, a fraction* $\mu_0 > 0$ *with bliss points at* $t = 0$ *and a fraction* $\mu_+ > 0$ *of the population with bliss points* $t > 0$ *with* $\mu_- + \mu_0 + \mu_+ = 1$*. To decentralize* $q = 0$*, we set* $p_- < 0$*,* $p_0 = 0$ *and* $p_+ > 0$ *such that* $\mu_- p_- + \mu_+ p_+ = 0$*. Clearly, such prices exist. Then for each agent with* $t < 0$ *we have a budget set* $qp_- \leq w_t = 0$ *which implies the set of affordable policies is* $Q \cap \mathbb{R}^+$*. Clearly, on such a set, the optimal choice is* 0*. Similarly, for agents with* $t > 0$ *we have a budget set* $qp_+ \leq w_t = 0$*, so the set of affordable policies is* $Q \cap \mathbb{R}^-$ *with an optimal choice of* 0*. Finally, agents* $t = 0$ *choose their bliss point* 0*.*

We summarize this with the following proposition. Its proof is straightforward given the above example.

Proposition 2. *Let Assumption 1 be satisfied. Let $q \in Q$ be such that there exists a strictly positive fraction of types with bliss points $t < q$ and a strictly positive fraction of types with bliss points $t > q$. Then there exists a Lindhal equilibrium with transfers decentralizing q.*

A corner/boundary policy may not be possible to be decentralized in a Lindhal equilibrium unless there is a (strictly) positive fraction of types with that policy as their bliss point.

Example 5 (Corner bliss-points)**.** *Consider a set of bliss points $[0, 1] = Q$ with a positive mass of agents at $t = 0$ (with mass $\mu_0 \in (0, 1)$) and the remaining measure of agents' ($\mu_+ = 1 - \mu_0$) distributed over $(0, 1]$. We set prices $p_+ > 0$ for agents with $t > 0$ but for $t = 0$ we set up $p_0 = -\frac{p_+\mu_+}{\mu_0}$. Again, with incomes $w_t = 0$ for each t, agents with $t > 0$ can afford only nonpositive q while agents $t = 0$ can afford any nonnegative q, and therefore both groups choose $q = 0$. The prices sum up to 0 and therefore the firm is indifferent and chooses $q = 0$ making zero profit. The sum of transfers is trivially zero. Alternatively, setting $p_0 < -\frac{p_+\mu_+}{\mu_0}$, agents at $t = 0$ choose their bliss points. Prices do not sum up to 0 but to some negative number. The firm chooses a lower bound of Q, $q = 0$.*

3.2 Multi-dimmensional Euclidian or Black type preferences

We start with the following definition.

Definition 3 (Balancedness)**.** *An $n \times m$ matrix A of real numbers is balanced if there exists a column vector $\alpha \in \mathbb{R}^m_{++}$ such that $A\alpha = 0 \in \mathbb{R}^n$.*

As we show below, the results for a single dimmentional Q cannot be extended in a straightforward way to a multi-dimentional Q, unless a certain matrix we define below is *balanced.* In fact, this *balancedness* condition is trivially satisfied for interior allocations in a convex hull of agents' bliss points in $Q \subset \mathbb{R}$.

We impose the following assumption:

Assumption 2. *Suppose Q is open and convex and u_t are Euclidian preferences with t denoting the bliss point of type t.*

We are now in a position to state the following proposition which establishes a key existence result:

Proposition 3. *Assume 2 and suppose $q \in Q$, and let $(p_t)_{t\in T} \in \mathbb{R}^L$ be a vector of personalized prices supporting q, i.e. $p_t \in \partial u_t(q)$. If the $L \times T$ matrix $[p_t]_{t\in T}$ is balanced, then there exists a Lindhal equilibrium with transfers decentralizing q.*

Proof. Since $[p_t]_{t\in T}$ is balanced, there exists a vector $(\alpha_t)_{t\in T}$ such that $\sum_{t\in T} \alpha_t p_t = 0$. Defining $p'_t = p_t \frac{\alpha_t}{\mu_t}$, we obtain $\sum_{t\in T} \mu_t p'_t = 0$. Setting $w'_t = p'_t \cdot q$, we obtain a Lindhal equilibrium $q, (p'_t)_{t\in T}$ with transfers $(w'_t)_{t\in T}$. In fact, as $\sum_{t\in T} \mu_t p'_t = 0$, the firm is indifferent and can choose $y^*_t = q, t \in T$. Transfers are also balanced. By construction, each t at prices p'_t and income w'_t can afford q. By definition $p'_t = \frac{\alpha_t}{\mu_t} p_t \in \partial u_t(q)$, and therefore q is the optimal choice of agents. □

For a finite set $A \subset \mathbb{R}^L$ define $\mathrm{con}^\circ(A)$ as the strict convext hull of $A \subset \mathbb{R}^L$, i.e.

$$\mathrm{con}^\circ(A) := \left\{ z \in \mathbb{R}^L : \forall a \in A\, \exists \alpha_a > 0 \text{ s.t. } \sum_{a\in A} \alpha_a = 1 \text{ and } z = \sum_{a\in A} a\alpha_a \right\}.$$

We interpret a policy contained in the interior of the strict convex hull of voter's bliss points as a compromise between the most preferred policy choices of different voter types.

Proposition 4. *Assume 2, let $q \in Q$ be such that $q \in con^\circ(T)$. Then there exists a Lindhal equilibrium with transfers decentralizing q.*

Proof. For each type in T define $p_t = t - q$. We will show that the matrix of prices $[p_t]_{t\in T}$ is balanced. In fact, since q is an element of $\mathrm{con}^\circ(T)$, we have $q = \sum_{t\in T} \alpha_t t$ with all $\alpha_t > 0$. Then

$$0 = \sum_{t\in T} \alpha_t t - q = \sum_{t\in T} \alpha_t [t - q] = \sum_{t\in T} \alpha_t p_t.$$

Hence $[p_t]_{t\in T}$ is balanced, therefore, by Proposition 3 $(y_t = q, \alpha_t p_t : t \in T)$ is a Lindhal equilibrium with transfers $w_t = \alpha_t p_t \cdot q$, $t \in T$. □

Hence, any Condorcet winner in $\text{con}^\circ(T)$ (so that it is a compromise policy) can be decentralized as a Lindahl equilibrium with transfers. The next example illustrates that the interiority assumption in the statement of the Proposion 4 is critical.

Example 6 (Nonexistence at the boundary)**.** *Consider $Q \in \mathbb{R}^2$ and three agent types of equal measure one with bliss points: one in $t_1 = (0,1)$, one in $t_2 = (-1,0)$ and one in $t_3 = (1,0)$, respectively. The policy $q = (0,0)$ belongs to the convex hull of the bliss points. However, a matrix of supporting prices $[p_t]_{t=1,2,3} = [(0,1),(-1,0),(1,0)]$ is not balanced and there is no Lindhal equilibrium decentralization q. However, there exists a Lindahl quasi-equilibrium decentralizing q with $(p_t)_{t=1,2,3} = ((0,0),(-1,0),(1,0))$ and incomes $w_1 = w_2 = w_3 = 0$.*

3.3 Further Results

3.3.1 Can any Condorcet be balanced?

Suppose Condorcet winner $q \in \mathbb{R}^L$ exists for a m-majority rule. Consider the vector of supporting prices $(p_t)_{t \in T}$ such that $p_t \in \partial u_t(q)$. Is any such matrix $[p_t]_{t \in T}$ balanced? The answer is no, as demonstrated by the following example.

Example 7. *Similarly to example 6, consider $Q \subset \mathbb{R}^2$ with $\mu_{t_1} = 0.1$ of agents in $t_1 = (0,1)$ and $\mu_{t_2} = \mu_{t_3} = 0.45$ of agents in $t_2 = (-1,0)$ and $t_3 = (1,0)$, respectively. The policy $q = (0,0)$ belonging to the convex hull of the bliss points $\{t_1, t_2, t_3\}$ is a m-majority winner for $m > 0.55$, i.e., any counterproposal will not collect a m-majority of the voters to beat q. However, the matrix of supporting prices $[p_t]_{t=1,2,3} = [(0,1),(-1,0),(1,0)]$ is not balanced. Nevertheless, there exists a quasi-equilibrium of Lindhal, with $(p_t)_{t=1,2,3} = ((0,0),(-1,0),(1,0))$ with income $w_1 = w_2 = w_3 = 0$.*

3.3.2 Condorcet winner and Plott (1967)

In the following example, we investigate the relations between our balancedness condition and the condition C proposed by Plott (1967), which we call pairwise balancedness.

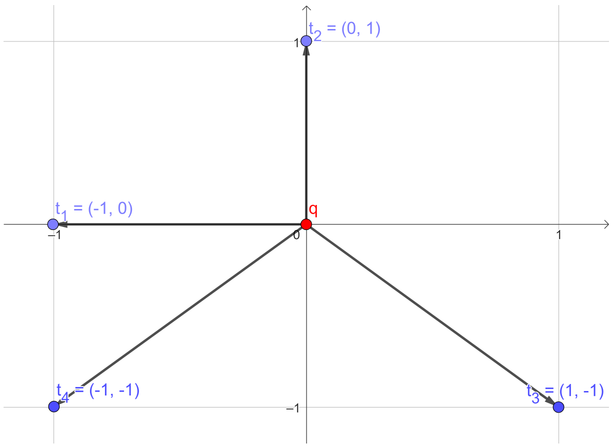

Figure 2: Consider 4 voters of equal measure with single peaked preferences with bliss points: $(-1,0),(0,1),(1,-1),(-1,-1)$. Proposal $q=(0,0)$ is balanced but not pairwise balanced.

Example 8 (Balancedness vs. Pair-wise Balancedness)**.** *Consider 4 individuals with single peaked preferences over $Q \subset \mathbb{R}^2$. Assume at a given point $q \in Q$ the gradients of individual policies form a matrix:*

$$\nabla U(q) = \begin{bmatrix} -1 & 0 & 1 & -1 \\ 0 & 1 & -1 & -1 \end{bmatrix}$$

as illustrated in the Figure 2. Observe that $\nabla U(q)$ is balanced. Indeed a vector of weights $\alpha = [1\,3\,2\,1]^T$ satysfies: $\nabla U(q)\alpha = 0$. But $\nabla U(q)$ cannot be divided into balanced pairs, as required by Plott (1967). Indeed, neither of the pairs including the first indyvidual is balanced:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}.$$

And hence balancedness does not imply condition C. However, the reverse implication is satisfied. If the number of agents is even and $\nabla U(q)$ is pairwise balanced, then it is trivially balanced. Our balancedness condition is therefore weaker than the condition C of Plott (1967).

3.3.3 First and Second Welfare Theorems

Our results can be interpreted and compared to the standard 1st and 2nd welfare theorems for production economies. The first welfare theorem says that under Assumption 2, if $q^*, (p^*_t)_{t\in T}$ is a Lindhal equilibrium with transfers, then q^* is Pareto optimal. To see that, argue by contradiction. Suppose that there exists $q' \in Q$ such that $u_t(q') \geq u_t(q^*)$ with strict inequality for at least one t. Then by consumer maximization in a Lindhal equilibrium: $p^*_t \cdot q' \geq w_t$ with strict inequalities for at least some t. Then

$$\sum_t \mu_t q' \cdot p^*_t > \sum_t \mu_t w_t = (\sum_t \mu_t p^*_t) \cdot q^* \geq (\sum_t \mu_t p^*_t) \cdot q',$$

where we first use condition 4 and 2 from the definition of a symmetric Lindhal equilibrium. This gives a contradiction.

Observe that under Assumption 2 Pareto optimality means that $q^* \in Q$ is in the convex hull of the set of agents' bliss points. Proposition 4 is our version of the second welfare theorem. Every Pareto optimal q can be decentralized as a quasi-equilibrium with transfers, and the "interior" q can be decentralized as an equilibrium with transfers.

3.3.4 Quasi-linear Utility and Non Zero Marginal Costs

Our results can be extended to incorporate quasilinear preferences and non-zero marginal costs of production. This extension can be useful if the production of "policies" or "social states" q is costly. We illustrate it for $L = 2$. Suppose that the preferences are given by $U_t : Q \times \mathbb{R} \to \mathbb{R}$ with $U_t(q, x_t) = u_t(q) + x_t$. The production function transforms $z \in \mathbb{R}$ to $q \in Q$ via $q = f(z)$. In this example, we assume $f(z) = \beta z$ with $\beta > 0$. Suppose that the economy is endowed with ω (distributed according to $(\omega_t)_{t\in T}$) units of the quasi-linear good while the decrease of q is zero.

Observe that for our results to hold in this economy we have to modify our balancedness condition to account for a quasilinear component (that is, that the preferences of all agents are monotone with a quasilinear good). The Lindhal equilibrium with transfers $(w_t)_{t\in T}$ is $(q^*, z^*, (p^*_t, p'_t)_{t\in T})$ such that for each type t, (q^*, z^*) solves $\max_{q,z} u_t(q) + \omega_t - z$

under $p_t^* q \leq p_t' z + w_t$, while (q^*, z^*) also solves $\max_{q,z}(q \sum_t \mu_t p_t^* - z \sum_t \mu_t p_t')$ and markets clear $\frac{q^*}{\beta} = z^* \leq \omega$ with $\sum_t \mu_t w_t = 0$. For concave and differentiable u_t to decentralize $q^* \leq \beta\omega$, one can set $p_t := u_t'(q^*)$. For a matrix of such prices $[p_t]_{t\in T}$, the appropriately adopted version of the balancedness conditions means that there exists a vector of strictly positive weights $(\alpha_t)_{t\in T}$ such that $\sum_t \alpha_t p_t = \frac{1}{\beta} \sum_t \alpha_t$. Under this condition, setting prices $p_t^* = \frac{\alpha_t}{\mu_t} p_t$ and $p_t' = \frac{\alpha_t}{\mu_t}$ implies a zero profit condition, and setting $w_t = z^*(\beta p_t^* - p_t')$ guarantees that agents can afford (q^*, z^*). Such transfers are balanced, i.e. $\sum_t \mu_t w_t = 0$, again, by a zero-profit condition.

Note that, to obtain the above result, we have allowed two personalized prices, that of q and that of a quasi-linear good. When normalizing all prices of the quasilinear good to the same number, e.g. setting $p_t' = 1$ for all $t \in T$ the equilibrium may not exist, as it may not be possible to balance $[p_t]_{t\in T}$.

4 Conclusion

This paper has proposed a new condition, balancedness, to decentralize Condorcet winners as Lindahl equilibria when voters' preferences are characterized by bliss points over social states, or equivalently, public policies. We have shown that any Condorcet winner located within the interior of the convex hull of voters' bliss points can be decentralized as a Lindahl equilibrium with transfers.

Our analysis examines the conditions under which market mechanisms can be used to align the interests of a dissenting minority in democratic settings. When our condition for balancedness is satisfied, in various contexts such as the "ideal" provision of public goods (e.g., a local park funded by subscription) or the "optimal" avoidance of public bads (e.g., the location of a municipal garbage dump), as well as the older literature on the market for votes (see Casella et al. (2012)), the prices and transfers derived from this condition can be interpreted as the subsidies required for minority voters who disagree with the majority choice. However, when a policy lies outside the interior of the convex hull of bliss points, meaning it is not a compromise between the most preferred policies of different voter types, the examples we construct suggest a limit to the use of market mechanisms to manage political disagreements.

An implicit assumption in the analysis presented in this paper is that all agents affected by policy choices are simultaneously present to vote. In practice, this assumption may not hold. For example, policies related to emissions mitigation affect not only the current generation but also future generations, who are not present today when policies are voted on. In such a setting, personalized prices calculated using our balancedness condition become of interest, as they can be used to determine intergenerational subsidies or transfers needed to decentralize Pareto-optimal climate policies.

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