arXiv:2406.14198v1 [econ.TH] 20 Jun 2024

Guaranteed shares of benefits and costs

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June 2024

Abstract

In a general fair division model with transferable utilities we discuss *endogenous* lower and upper guarantees on individual shares of benefits or costs. Like the more familiar *exogenous* bounds on individual shares described by an outside option or a stand alone utility, these guarantees depend on my type but not on others' types, only on their number and the range of types. Keeping the range from worst share to best share as narrow as permitted by the physical constraints of the model still leaves a large menu of tight guarantee functions.

We describe in detail these design options in several iconic problems where each tight pair of guarantees has a clear normative meaning: the allocation of indivisible goods or costly chores, cost sharing of a public facility and the exploitation of a commons with substitute or complementary inputs. The corresponding benefit or cost functions are all sub- or super-modular, and for this class we characterise the set of minimal upper and maximal lower guarantees in all two agent problems.

JEL classification codes: D6, D63

1 Introduction

Nash axiomatic bargaining model ([23]) and much of the work it inspired acknowledge the importance of outside options, aka disagreement points, critically limiting the range of mutually advantageous outcomes where the

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‡We are grateful for the comments we received in WINE 2023 at Shanghai Tech University, SAET 2024 at the University of Santiago, and Game Theory Days at UM6P Rabat; as well as in seminars at the Universities of Paris 1, Vienna , Wuhan and Surrey. Special thanks to Miguel Ballester, Yuqing Kong, Dhivya A. Kumar and Bo Li.

negotiation is headed. Steinhaus' cake cutting model ([29]), another early fair division model, sends a similar message: we can always give to each agent a piece of the cake worth at least $\frac{1}{n}$-th of its total value to them, so everyone's actual share should be at least as good as this fair share.

In both cases the allocation of common property resources (the cake, the outcomes they must jointly agree upon) is restricted by entitling each agent to a specific guarantee, but there is a key difference: an outside option in bargaining is an *exogenous default* this agent can exercise by herself, while a fair share of the cake is an *endogenous guarantee* that she cannot implement independently of others. A benevolent manager is needed to enforce the latter, e. g. by playing a moving knife[1] ([8]) or a divide and choose ([13]) game; the threat to engage in this severely inefficient mechanism has the same deterring effect than an exogenous outside option.

The difference between the two types of wost case guarantee is especially clear in the celebrated problem of the commons ([27], [26], [28]) where the agents enter inputs in a common property production function. For instance they pull their purchases to take advantage of a volume discount: the option to "stand alone" at the non-discounted price is an exogenous upper bound on my cost share if the seller accepts orders of small size. But a benchmark price recognised by all participants can play the same role, even if small orders are not currently available.

Endogeneity opens the door to design options in the many problems where instead of a single *optimal* (largest) fair share the manager can choose from a whole menu of *maximal* fair shares (that cannot be improved): this choice is the object of our paper.

My fair share of the resources defines my *worst case* scenario, which is why we want it large. But the dual concern to keep my *best case* scenario low is important as well, lest other agents will object because it is too large. If the commons produces output with decreasing returns,[2] the *stand alone* share I would get by using the commons all by myself is a compelling upper bound on my share: any larger share forces the other agents to work for me ([17], [11]).

A first normative step toward interpreting common property is, in the view presented here, to stake out ex ante the range of each agent's own share as a function of their own type (personal characteristics such as willingness to pay, input effort, demand of output etc..) but not of other participants'

[1]The knife cuts an increasing share of the cake and whoever stops the knife first receives that share; repeat.

[2]as opposed to the increasing returns of the bulk purchasing example.

types. Making this range as small as permitted by the feasibility constraints of the resources facilitates agreement on an ex post compromise achieved after direct negotiations, a fully scripted mechanism, or any other decision making format.

In our abstract model of dividing benefits or costs, searching for the tightest pairs of individual guarantees from below and above has a transparently simple mathematical formulation as a functional inequality, given in the next paragraph. There are two surprises. First, solving this set of inequalities is quite hard and was within our reach only for relatively simple versions of familiar fair division models and some new variants: the allocation of indivisible goods or costly chores and cash transfers, cost sharing of a public facility and the exploitation of a commons with substitute or complementary inputs. Second, in these examples each solution has a rich normative meaning easily explained from the components of the model. Some solutions confirm the relevance of familiar contradictory interpretations of common property; others propose appealing compromises between these extreme views, or new guarantees in new problems.

We illustrate these points in section 2 by solving the inequalities for perhaps the simplest cost or benefit function in the literature, interpretable as sharing either the cost of a capacity, or one indivisible item with cash transfers.

the functional inequality The general model has n agents, each with their own type x_i varying in a common domain $\mathcal{X}$, and a symmetric benefit (or cost) function $\mathcal{W}$: from a profile of types $x = (x_1, \cdots, x_n)$ it returns the total benefit (or cost) $\mathcal{W}(x)$ that will be divided between the n agents. The technology $\mathcal{W}$ is the common property of the agents, all responsible for their own type in $\mathcal{X}$.

A *lower guarantee* g^- and an *upper guarantee* g^+ are real valued functions on $\mathcal{X}$ such that:

$$\sum_1^n g^-(x_i) \leq \mathcal{W}(x) \leq \sum_1^n g^+(x_i) \text{ for any } x = (x_1, \cdots, x_n) \in \mathcal{X}^{[n]} \quad (1)$$

The high level mathematical goal is to find all solutions g^-, g^+ of these inequalities. This amounts to describe in closed form the *maximal lower* (resp. *minimal upper*) guarantees, those that cannot increase (resp decrease) at any value x_1 without violating (1) for some choice of $x_2, \cdots, x_n$. They are the tightest additively separable approximations of the function $\mathcal{W}$ from below and above. To the best of our knowledge, this mathematical question is original.

fair shares in the literature After its appearance in the mathematical cake cutting model, the concept of endogenous fair share played a central role in the normative microeconomic literature of the allocation of common property private commodities. There the ordinal version of the fair share guarantee is the *equal split* welfare: that of consuming the physical $\frac{1}{n}$-th share of the bundle we divide. It is the virtual initial endowment of the competitive solution favored by microeconomists ([32], [31], [3], [22]). Importantly these two definitions of the fair share are an instance of the general concept of *unanimity* welfare or utility: the one I get if everyone else has exactly the same preferences (or more generally, characteristics) as me and we are treated equally (a minimal fairness requirement in that case). This very concept is central throughout the paper.

We note that equal split defines the optimal fair share in these two models only if preferences are convex in the Arrow Debreu model or utilities are additive over the subsets of the cake. With monotone but non convex preferences in the Arrow-Debreu model equal split still captures a maximal lower guarantee, but it is not the only one and we do not know a single other example; for a non atomic cake with general (non additive) utilities, not even one maximal lower guarantee is known ([4]).

When the shared resource is a technology the first concept of fair share was the stand alone utility mentioned above with its versatile role as a lower or upper bound on welfare depending upon the returns to scale . The unanimity welfare was not far behind, also switching roles between worst and best case welfare for the same reasons ([27], [18], [19], [20], [10]).

In the last twenty years or so, a large stream of research at the interface of computer science and microeconomics addresses the deterministic fair division of indivisible items (good or bad) with no cash compensations. Even when utilities are additive over bundles of goods, the definition of a meaningful fair share is a tough conceptual challenge still attracting considerable attention. The first proposal of the *MaxMinShare* is an elegant adaptation of the unanimity concept to the constraints: it is the utility of my worst share in the best n-partition of these objects I can offer ([6]). But it is hard to compute and also unfeasible in some (very) rare utility profiles (([25])). Alternative concepts include a blunt $\frac{3}{4}$ fraction of the MaxMinShare, always feasible (([1])); the *any price share* ([2]) still occasionally unfeasible but much easier to compute than the MaxMin Share; Hill's bound, both feasible and easy to compute, when we can upper bound the relative weight of any item ([12], [7], [16], [14]) and more.

Contrasting with the above streams of research, our model is mathematically (much) simpler and more versatile: our examples include the allocation of indivisible private items (good or bad), cost sharing of a capacity or a public facility, and several versions of the commons problem. But we rely heavily on the assumption, typically absent in the literature just reviewed, that utility is transferable via some numeraire like cash payments.

Contents Section 2 illustrates our methodology for the canonical function $\mathcal{W}(x) = \max_{1\leq i\leq n}\{x_i\}$ with two alternative interpretations as the allocation of one indivisible object for which i is willing to pay x_i, or sharing the cost of a public capacity when x_i is (the cost of) agent i's need. Proposition 1 gives the full solution of inequalities (1).

Section 3 introduces the general model; the key definition of the *contact set* of an extremal (maximal lower or minimal upper) guarantee: the set of profiles x at which inequality (1) is an equality; and two critical Lipschitz and differentiability properties that such guarantees inherit from the function $\mathcal{W}$ (Lemmas 6,7).

Section 4 restricts attention to super (resp. sub) modular benefit or cost functions. The Unanimity guarantee $una(x_i) = \frac{1}{n}\mathcal{W}(x_i, x_i, \cdots, x_i)$ is the smallest upper (resp. largest lower) guarantee (Proposition 2), which completely solves one side of the system (1). Moreover these functions admit two canonical maximal lower (resp. minimal upper) *incremental* guarantees (Proposition 3). In the commons problem we find $(n-2)$ additional extremal guarantees of the same incremental type (Proposition 4), compromising between the two canonical ones.

Sections 5 solves system (1) for the rich class of *rank-separable* functions $\mathcal{W}$ of the form $\mathcal{W}(x) = \sum_{k=1}^{n} w_k(x^k)$ where $(x^k)_1^n$ is the order statistics of $(x_i)_1^n$. Theorem 1, a considerable generalisation of Proposition 1, describes the $(n-1)$-dimensional set of extremal guarantees for rank-separable and modular functions $\mathcal{W}$. Examples of such functions include sharing the cost of connecting agents located on a line, sharing the output produced by the median effort in the group, and other variations of the commons problem.

Theorem 2 in section 6 solves system (1) for all *two agent* problems with a (strictly) super- or sub-modular function $\mathcal{W}$. The dimension of the set of solutions is infinite.

The difficulties toward solving system (1) with three or more agents are illustrated in subsection 7.1 for the standard version of the commons problem. Whether or not our extremal guarantees respect the decentralisation of a benefit or cost in two independent sub-problems is discussed in subsection 7.2 for the allocation of multiple objects with additive utilities.

After some final take-home points in section 8, section 9 collects several long proofs.

2 A canonical example

We describe the pattern of solutions of (1) in two of the simplest and most familiar fair division problems formalised by the same function $\mathcal{W}$.

sharing the cost of a capacity ([15]) The n agents share a public facility (canal, runway...) adjusted to their different needs (for a canal more or less wide or deep, for a short or long runway...). The cost of building enough capacity to serve the needs of agent i is x_i (we identify a need and its cost); the cost of serving everyone, $\max_{i\in[n]}\{x_i\}$, must be divided in n shares y_i s. t. $\sum_i y_i = \max_{i\in[n]}\{x_i\}$. The largest possible range of individual needs x_i is the interval $[L, H]$ where $0 \leq L < H$.

assigning an indivisible good or bad The n agents must assign an indivisible object that could be desirable (a good) or not (a bad, e. g., a chore). Utilities are linear in money and described by a single real number x_i: if x_i is positive agent i is willing to pay that much for the object, if x_i is negative our agent must be compensated at least $|x_i|$ to accept the object (do the chore). Utilities vary in the real interval $[L, H]$, so if $L < 0 < H$ the object can be a good for some agents and a bad for others.

For efficiency we must assign the object to an agent i s. t. $x_i = \max_{j\in[n]}\{x_j\}$. If agent j does not get the object and $x_j > 0$ she deserves some cash compensation from the efficient agent who gets it; if $x_i < 0$ and does the chore she definitely deserves some compensation because everyone else would suffer even more to do it. In all cases the amount $\max_{j\in[n]}\{x_j\}$ (a benefit or a cost) will be divided in n shares y_i: a cash transfer if agent i does not get the object, and $y_i = x_i - \sum_{j\neq i} y_j$ if she does.

extremal guarantees Notice that "best share" means smallest cost share y_i in the capacity story but largest utility y_i in the assignment story. Lower and upper guarantees g^- and g^+ are real valued functions on $[L, H]$ satisfying the system

$$\sum_{i=1}^{n} g^{-}(x_i) \leq \max_{1\leq i\leq n}\{x_i\} \leq \sum_{i=1}^{n} g^{+}(x_i) \text{ for all } x \in [L,H]^n \tag{2}$$

We call the lower guarantee g^- *maximal* if we cannot increase it at any single value x_1 without violating some left hand inequalities (LH) in (2). There is in fact only one such function g^- and it is very simple.

The *Unanimity share* $una(x_1) = \frac{1}{n}x_1$ is agent 1's fair share of $\mathcal{W}(x)$ at the unanimous profile where $x_j = x_1$ for all j. At such profile (2) implies $g^-(x_1) \leq una(x_1)$; moreover $una(\cdot)$ itself is a lower guarantee (the mean is smaller than the max), therefore una is the largest of *all* lower guarantees, and the only maximal one.

By contrast we show below that there is an infinite (one-dimensional) set of minimal upper guarantees. The two end points of this set the Stand Alone g^+_{sa} and Egalitarian g^+_{ega} guarantees:

$$g^+_{sa}(x_i) = x_i - \frac{n-1}{n}L \text{ and } g^+_{ega}(x_i) = \frac{1}{n}H \text{ for all } x_i \in [L,H]$$

(we let the reader check that they meet the RH in (2))

The stand alone terminology is clear if $L = 0$: $g^+_{sa}(x_i)$ is the cost of the capacity that i needs, or the utility of consuming the good object and pay nothing to others. For other values of L we note that $g^+_{sa}(L) = \frac{1}{n}L = una(L)$, so the Stand Alone guarantee gives exactly this share to any agent with the smallest type (need or willingness to pay). Therefore at other types x_i the upper guarantee $g^+_{sa}(x_i)$ is i's share if everyone else is of type L and i is standing alone.

To see that g^+_{sa} is minimal fix an arbitrary x_1 and select $x_j = L$ for all $j \geq 2$: as these agents all get the share $\frac{1}{n}L$ the LH in (2) implies that agent 1 gets at least $g^+_{sa}(x_1)$.

The egalitarian upper guarantee $g^+_{ega}(x_i) = \frac{1}{n}H$, together with the unanimity lower bound, ensures that an agent with the largest type H gets exactly $\frac{1}{n}H$. Hence an agent i with a lower type x_i must pay exactly $\frac{1}{n}H$ if at least other agent is of type H: no one gets more than $\frac{1}{n}H$ and H must be distributed, therefore agent 1 must get that share too. In particular this implies that g^+_{ega} is minimal.

The normative choice between g^+_{sa} and g^+_{ega} is stark. The stand alone guarantee is easier on agents who need a small capacity than the egalitarian one, and vice versa for agents with large needs. The opposite pattern holds for the assignment of a good.

Proposition 1: *With the notation $z_+ = \max\{z, 0\}$, the minimal upper guarantees g_p^+ of $\mathcal{W}(x) = \max_{1 \leq i \leq n}\{x_i\}$ are parametrised by $p \in [L, H]$ as follows:*

$$g_p^+(x_i) = \frac{1}{n}p + (x_i - p)_+ \text{ for all } x_i \in [L, H] \tag{3}$$

where $g_L^+ = g_{sa}^+$ and $g_H^+ = g_{ega}^+$.

Proof: Assume g^+ is a minimal upper guarantee and set $p = ng^+(L)$. At the unanimous profile $x_i \equiv L$ the RH in (2) implies $p \geq L$. Minimality implies that g^+ increases weakly (Lemma 3 in section 3) so that $g^+(x_i) \geq \frac{1}{n}p$ for all x_i; moreover if $p > H$ we $g^+ > g_{ega}^+$ everywhere and g^+ is not minimal, hence $p \leq H$.

Inequality (2) applied to x_i and $n-1$ agents with utility L gives $g^+(x_i) \geq x_i - \frac{n-1}{n}p$; combining this with $g^+(x_i) \geq \frac{1}{n}p$ gives $g^+ \geq g_p^+$. To check finally that g_p^+ meets the right inequalities in (2) is routine. ■

Interpret the upper guarantee g_p^+ to share the cost of a capacity: if $x_i \leq p$ agent i pays at least $\frac{1}{n}x_i$ and at most $\frac{1}{n}p$ in the worst case. An instance of the latter is when only one agent i^* needs a capacity larger than p; this is also the worst case for i^* who pays, in addition to $\frac{1}{n}p$, the full incremental cost $x_{i^*} - p$.

Interpret the guarantee g_p^+ to assign an indivisible object: p could be the market price for a good (if $0 < p < H$) or a reference wage for performing the chore (if $L \leq p < 0$).

If the object is a good the inefficient agent i whose utility is below p can get a benefit share of at most $\frac{1}{n}p$, while an efficient agent with $x_i \geq p$ can get at most $x_i - \frac{n-1}{n}p$. Both upper bounds are reached if an efficient agent has $x_i \geq p$ while $x_j \leq p$ for everyone else.

If we assign a bad an inefficient agent s. t. $x_i \leq p$ pays at least $\frac{1}{n}|p|$ and at most $\frac{1}{n}|x_i|$. But the efficient agent with a disutility for the chore below the benchmark, $|x_i| < |p|$, can be paid up to $\frac{n-1}{n}|p|$ so she may end up with a net profit.

On Figure 1 it is clear that a (true) convex combination of g_{sa}^+ and g_{ega}^+ is clearly another upper guarantee, however it is never minimal. One checks easily that for any $\lambda \in]0, 1[$ we have

$$\{\lambda g_{sa}^+ + (1 - \lambda)g_{ega}^+\}(x_1) > g_{\lambda L+(1-\lambda)H}^+(x_1) \text{ for all } x_1 \in]L, H[$$

The Corollary to Lemma 7 generalises this observation.

selecting a mini-maximising or mini-averaging pair of guarantees Two natural selections in the family $\{g_p^+; L \leq p \leq H\}$ are the one minimising the maximal spread (over types) between the two share levels, or the average spread over a given distribution of types; they solve respectively the two programs

$$\min_p \max_{x_1}\{g_p^+(x_1) - una(x_1)\} \text{ and } \min_p \int_L^H (g_p^+(t) - una(t))dt$$

(in the latter with the uniform distribution). The mini-maximising selection is easily computed as $p = \frac{1}{n}L + \frac{n-1}{n}H$ and the mini-averaging one is $p = \frac{n}{2n-1}L + \frac{n-1}{2n-1}H$ close to the midpoint of $[L, H]$.

implementing the guarantees We do not discuss the role of the benevolent manager after the choice of a pair of guarantees. If all information about individual types is public (as is typical in this section's cost sharing story, as well as the commons and facility location examples in sections 4 and 5) she can either enforce a deterministic division rule to distribute $\mathcal{W}(x)$ among the n agents, or she can let the agents engage in unscripted face to face negotiations for which the lower and upper guarantees set clear constraints on the final outcome. In both scenarios and everything in between, choosing tight bounds on individual shares promotes participation: to the mechanism by minimising the risk of playing badly, or to direct negotiations by lowering the impact of being manipulated by agressive negotiators.

If types can be private information (as in this section's assignment story and Proposition 5 section 7) then an agent well informed about other participant's willingness to pay and predictable behavior after the guarantees are in place can benefit by misreporting his type. The only agents from which we expect a truthful report are those without such information and unwilling to take risks: any over-reporting (resp. under-reporting) of my type could (for certain configurations of other reported characteristics) hurt me if I end up getting the object (resp. not getting it).

3 General model

The set of agents is $[n] = \{1, \cdots, n\}$ and $\mathcal{X}$ is their common set of relevant types. A profile $x = (x_i)_{i\in[n]} \in \mathcal{X}^{[n]}$ generates the efficient benefit or cost $\mathcal{W}(x)$, where $\mathcal{W}$ is a *symmetric* real valued function of its n variables x_i. We assume throughout that $\mathcal{X}$ is a compact metric space and $\mathcal{W}$ is continuous.

Benefits or costs are transferable by cash payments or some other numeraire. The variable y_i is agent i's net (dis)utility after cash transfers. The (dis)utility profile $y = (y_i)_{i\in[n]}$ is feasible *iff* $\sum_{i=1}^n y_i \leq \mathcal{W}(x)$ (the opposite inequality for disutility) and efficient *iff* this is an equality.

3.1 lower and upper guarantees

Definition 1: *The functions g^- and g^+ from $\mathcal{X}$ into $\mathbb{R}$ are respectively a lower and an upper guarantee of $\mathcal{W}$ iff they satisfy the inequalities*

$$\sum_{i=1}^{n} g^-(x_i) \leq \mathcal{W}(x) \leq \sum_{i=1}^{n} g^+(x_i) \text{ for all } x \in \mathcal{X}^{[n]} \tag{4}$$

We write $\boldsymbol{G}^+, \boldsymbol{G}^-$ the sets of such guarantees.

Given $g^1, g^2 \in \boldsymbol{G}^-$ (resp. $\boldsymbol{G}^+$) we say that g^1 dominates g^2 if $g^1(x) \geq g^2(x)$ (resp. $g^1(x) \leq g^2(x)$) for all $x \in \mathcal{X}$ and $g^1 \neq g^2$. The lower (resp. upper) guarantee $g \in \boldsymbol{G}^-$ (resp. $g^+ \in \boldsymbol{G}^+$) is *maximal* (resp. *minimal*) if increasing (resp. decreasing) its value at a single $x_1 \in \mathcal{X}$ creates a violation of the LH (resp. RH) inequality in (4) for some $x_{-1} \in \mathcal{X}^{[n-1]}$. We write $\mathcal{G}^-$ and $\mathcal{G}^+$ for the corresponding subsets of $\boldsymbol{G}^-$ and $\boldsymbol{G}^+$, and refer to $\mathcal{G}^- \cup \mathcal{G}^+$ as the set of *extremal* guarantees of $\mathcal{W}$.

Lemma 1 *For $\varepsilon = +, -$ every guarantee $g \in \boldsymbol{G}^\varepsilon \diagdown \mathcal{G}^\varepsilon$ is dominated by an extremal one.*

The omitted proof is a simple application of Zorn's Lemma.

The restriction of $\mathcal{W}$ to the diagonal of $\mathcal{X}^{[n]}$ define the *unanimity* share of agent i:

$$una(x_i) = \frac{1}{n}\mathcal{W}(x_i, x_i, \cdots, x_i) \tag{5}$$

Its importance is clear once we observe that (4) implies for any $g^-, g^+ \in \boldsymbol{G}^- \times \boldsymbol{G}^+$

$$g^-(x_i) \leq una(x_i) \leq g^+(x_i) \text{ for all } x_i \in \mathcal{X} \tag{6}$$

Lemma 2
i) If una is a lower (resp. upper) guarantee it dominates each lower (resp. upper) guarantee

$$una \in \boldsymbol{G}^\varepsilon \Longrightarrow \mathcal{G}^\varepsilon = \{una\} \text{ for } \varepsilon = +, -$$

ii) The function una is both a lower and an upper guarantee if and only if $\mathcal{W}(x) = \sum_i una(x_i)$, i. e., $\mathcal{W}$ is additively separable.

The easy proof is again omitted.

3.2 topological properties

Lemma 3 *If $\mathcal{X}$ is ordered by $\succ$ and $\mathcal{W}$ is monotone, so is every extremal guarantee in $\mathcal{G}^\varepsilon$*, for $\varepsilon = +, -$.

Proof Fix $g \in \boldsymbol{G}^-$. If $x_i \succ x_i'$ and $g(x_i) < g(x_i')$ define $\widetilde{g}(x_i) = g(x_i')$ and $\widetilde{g} = g$ otherwise, then check that $\widetilde{g}$ is still in $\boldsymbol{G}^-$. Same argument for $\boldsymbol{G}^+$. ■

The next result is the operational characterisation of extremal guarantees.

Lemma 4 *Fix $\varepsilon = +, -$ and an equi-continuous function $\mathcal{W}$ in $\mathcal{X}^{[n]}$.*
i) An extremal guarantee in $\mathcal{G}^\varepsilon$ is continuous in $\mathcal{X}$.
ii) A guarantee g in $\boldsymbol{G}^\varepsilon$ is extremal if and only if

$$\forall x_1 \in \mathcal{X}\ \exists x_{-1} \in \mathcal{X}^{[n-1]} : \sum_{i=1}^{n} g(x_i) = \mathcal{W}(x_1, x_{-1}) \tag{7}$$

If equality (7) holds we call (x_1, x_{-1}) a contact profile of g at x_1 ; the set of such profiles is the contact set of g, written as $\mathcal{C}(g)$.

Proof *Step 1.* We fix $g \in \mathcal{G}^-$ and check that it is upper-hemi-continuous. If it is not there is in $\mathcal{X}$ some x_1, a sequence $\{x_1^t\}$ converging to x_1, and some $\delta > 0$ such that $g(x_1^t) \geq g(x_1) + \delta$ for all t. Then we have, for any $x_{-1} \in \mathcal{X}^{[n-1]}$

$$\mathcal{W}(x_1^t, x_{-1}) \geq g(x_1^t) + \sum_{i=2}^{n} g(x_i) \geq (g(x_1) + \delta) + \sum_{i=2}^{n} g(x_i)$$

Taking the limit in t of $\mathcal{W}(x_1^t, x_{-1})$ and ignoring the middle term we see that we can increase g at x_1 without violating (4), a contradiction of our assumption $g \in \mathcal{G}^-$.

Step 2. Statement ii) "If" is clear. For "only if" we fix $g \in \mathcal{G}^-$ and show that it meets property (7). For any $x_1 \in \mathcal{X}$ define

$$\delta(x_1) = \min_{x_{-1} \in \mathcal{X}^{[n-1]}} \{\mathcal{W}(x_1, x_{-1}) - \sum_{i=1}^{n} g^-(x_i)\}$$

and note that this minimum is achieved at some $\overline{x}_{-1}$ because the function $x_{-1} \to \sum_{i=2}^{n} g^-(x_i)$ is upper-hemi-continuous (step 1). Moreover $\delta(x_1)$ is non negative.

If $\delta(x_1) = 0$ property (7) holds at $\overline{x}_{-1}$. If $\delta(x_1) > 0$ we can increase g at x_1 to $g(x_1) + \delta(x_1)$, everything else equal, to get a guarantee dominating g.

Step 3. We fix $g \in \mathcal{G}^-$ and check that it is lower-hemi-continuous. We write d for the distance of the metric space $\mathcal{X}$. By assumption $\mathcal{W}$ is equi-continuous in its first variable, uniformly in the others:

$$\forall \eta > 0, \exists \theta > 0, \forall x_1, x_1^*, x_{-1} : d(x_1, x_1^*) \leq \theta \Rightarrow \mathcal{W}(x_1, x_{-1}) \leq \mathcal{W}(x_1^*, x_{-1}) + \eta \tag{8}$$

If g is not l.h.c. there is some x_1 and $\{x_1^t\}$ converging to x_1 and $\delta > 0$ s.t. $g(x_1^t) \leq g(x_1) - \delta$ for all t. Pick θ for which (8) holds with $\eta = \frac{1}{2}\delta$ and t large enough that $d(x_1^t, x_1) \leq \theta$: then for any x_{-1} we have

$$g(x_1) + \sum_{i=2}^{n} g(x_i) \leq \mathcal{W}(x_1, x_{-1}) \leq \mathcal{W}(x_1^t, x_{-1}) + \frac{1}{2}\delta$$

Replacing $g(x_1)$ with $g(x_1^t) + \delta$ gives $g(x_1^t) + \sum_{i=2}^{n} g(x_i) \leq \mathcal{W}(x_1^t, x_{-1}) - \frac{1}{2}\delta$ for any x_{-1}: this contradicts the contact property (7) for x_1^t. ■

Remark 1 The defining inequalities (4) imply at once that $\boldsymbol{G}^{\varepsilon}(\mathcal{W})$ is closed under pointwise convergence. The same is true of $\mathcal{G}^{\varepsilon}(\mathcal{W})$ by Lemma 4 and compactness of $\mathcal{X}$.

Lemma 5

i) For $\varepsilon = +, -$ and any $x_1 \in \mathcal{X}$ there is an extremal guarantee $g \in \mathcal{G}^{\varepsilon}$ s.t. $g(x_1) = una(x_1)$.
ii) The set $\mathcal{G}^{\varepsilon}$ is a singleton if and only if it contains una. Moreover both $\mathcal{G}^-$ and $\mathcal{G}^+$ are singletons if and only if $\mathcal{W}$ is additively separable.

Note that statement i) does not imply that the graph of *every* extremal guarantee g in $\mathcal{G}^{\varepsilon}$ touches the unanimity function at some x_1: in the commons problem (subsection 4.3) Proposition 5 describes n extremal guarantees of the incremental type of which $n-2$ never touch the unanimity graph; see also Theorem 1 in section 5.

Proof *Statement i*) Fix $\varepsilon = -$, an arbitrary $\widetilde{x}_1 \in \mathcal{X}$ and write $B(\widetilde{x}_1, r)$ for the closed ball of center $\widetilde{x}_1$ and radius r. Use the notation $\Delta(x) = \sum_1^n una(x_i) - \mathcal{W}(x)$ to define the function

$$\delta(x_1) = \max\{\Delta(x_1, x_{-1}) : x_i \in B(\widetilde{x}_1, d(x_1, \widetilde{x}_1)) \text{ for } i \geq 2\}$$

It is clearly continuous, non negative because $\Delta(x_1, x_{-1}) = 0$ if $x_i = x_1$ for all $i \geq 2$, and $\delta(\widetilde{x}_1) = 0$. Define $g = una - \delta$ and check that g is the desired lower guarantee of $\mathcal{W}$. At an arbitrary profile $x = (x_i)_1^n$ choose x_{i^*} s.t. $d(\widetilde{x}_1, x_{i^*})$ is the largest: this implies $\delta(x_{i^*}) \geq \Delta(x)$. Combining this with $\delta(x_i) \geq 0$ for all $i \neq i^*$ gives $\sum_1^n \delta(x_i) \geq \Delta(x)$ which, in turn, is the left hand inequality in (4) for g. As g is in $\boldsymbol{G}^-$, it is dominated by some $\widetilde{g}$ in $\mathcal{G}^-$(Lemma 1) and $\widetilde{g}(x_1) = una(\widetilde{x}_1)$ by inequality (6).

Statement ii) We assume that $\boldsymbol{G}^-$ does not contain *una* and check that $\boldsymbol{G}^-$ is not a singleton. This assumption and the continuity of $\mathcal{W}$ imply that for an open set of profiles $x \in \mathcal{X}^{[n]}$ we have $\sum_i una(x_i) > \mathcal{W}(x)$. Fix such an

x and (invoking Lemma 1) pick for each i a maximal guarantee g_i equal to *una* at x_i: these n guarantees are not identical.

The second part of statement *ii*) follows from statement *ii*) in Lemma 2. ■

3.3 the Lipschitz and differentiability properties

They are key to the characterisation results in sections 5, 6 and 7. If (x_1, x_{-1}) is a contact profile of g, we also say that x_{-1} is a contact profile of g at x_1.

Lemma 6 *Fix any $\varepsilon = +, -$ and g in $\mathcal{G}^\varepsilon$. For any x_1, x_1' and any contact profile x_{-1} of g at x_1 we have*
if $\varepsilon = +$

$$g(x_1') - g(x_1) \geq \mathcal{W}(x_1', x_{-1}) - \mathcal{W}(x_1, x_{-1}) \tag{9}$$

and the opposite inequality if $\varepsilon = -$.

Proof In the inequality

$$g(x_1') + \sum_{i=2}^{n} g(x_i) \leq \mathcal{W}(x_1', x_{-1})$$

we replace each term $g(x_i)$ by $\mathcal{W}(x_1, x_{-1}) - \sum_{j \neq i} g(x_j)$ and rearrange the resulting inequality to

$$(n-1)(\mathcal{W}(x_1, x_{-1}) - g(x_1)) + (n-2)\sum_{i=2}^{n} g(x_i) \leq \mathcal{W}(x_1', x_{-1}) - g(x_1')$$

$$\Longleftrightarrow \mathcal{W}(x_1, x_{-1}) - g(x_1) + (n-2)(\mathcal{W}(x) - \sum_{i=1}^{n} g(x_i)) \leq \mathcal{W}(x_1', x_{-1}) - g(x_1')$$

and the claim follows because x_{-1} is a contact profile for g at x_1. ■

Lemma 7
i) If K is a positive constant and $\mathcal{W}$ is K-Lipschitz in each x_i, uniformly in $x_{-i} \in \mathcal{X}^{[n-1]}$, so is each g in $\mathcal{G}^- \cup \mathcal{G}^+$.
ii) Suppose $\mathcal{X} = [L, H]$ is an interval in some $\mathbb{R}^A$ where A is a finite set. We fix $x_1 \in \mathcal{X}$, an extremal guarantee $g \in \mathcal{G}^- \cup \mathcal{G}^+$ and a contact profile $x_{-1} \in \mathcal{X}^{[n-1]}$ of g at x_1. If g and $\mathcal{W}(\cdot, x_{-i})$ are both differentiable in x_{1a} at x_1 for some $a \in A$, we have

$$\frac{dg}{dx_{1a}}(x_{1a}) = \frac{\partial \mathcal{W}}{\partial x_{1a}}(x_1, x_{-1}) \text{ if } L_a < x_{1a} < H_a \tag{10}$$

$$\frac{dg}{dx_{1a}}(x_{1a}) \leq \frac{\partial \mathcal{W}}{\partial x_{1a}}(x_1, x_{-1}) \text{ if } x_1 = L \text{ and } g \in \mathcal{G}^- \text{ or } x_1 = H \text{ and } g \in \mathcal{G}^+$$

$$\frac{dg}{dx_{1a}}(x_{1a}) \geq \frac{\partial \mathcal{W}}{\partial x_{1a}}(x_1, x_{-1}) \text{ if } x_1 = H \text{ and } g \in \mathcal{G}^- \text{ or } x_1 = L \text{ and } g \in \mathcal{G}^+$$

Proof *Statement i)* If $g \in \mathcal{G}^-$ property (9) and the Lipschitz assumption imply $g(x_1) - g(x_1') \leq K\|x_1 - x_1'\|$ (where $\|\cdot\|$ is the norm w. r. t. which $\mathcal{W}$ is Lipschitz). Exchanging the roles of x_1 and x_1' gives $g(x_1') - g(x_1) \leq K\|x_1' - x_1\|$ and the conclusion.

For statement *ii)* note that if the functions f, g of one real variable z are differentiable at some z_0 in the interior of their common domain and the inequality $f(z) - f(z_0) \geq g(z) - g(z_0)$ holds for z close enough to z_0, then their derivatives at z_0 coincide. Apply this to the functions $x_{1a} \to g(x_1)$ and $x_{1a} \to \mathcal{W}(x_1, x_{-1})$ and the inequalities (9). The last two inequalities are equally easy to check. ■

Note that in the real line the Lipschitz property in statement *i)*, that we call *uniformly Lipschitz* by a slight abuse of terminology[3], implies differentiability almost everywhere. All our examples in sections 5 and 7 involve functions $\mathcal{W}$ uniformly Lipschitz in this sense, therefore all corresponding extremal guarantees are differentiable almost everywhere in each coordinate of x_i.

Corollary *Suppose $\mathcal{X} = [L, H]$, $\mathcal{W}$ is differentiable in $[L, H]^{[n]}$ and for $\varepsilon = +, -$ the extremal guarantees in $\mathcal{G}^\varepsilon$ are a. e. differentiable. Then an extremal guarantee is characterised by its contact set $\mathcal{C}(g)$:*

$$\mathcal{C}(g) = \mathcal{C}(h) \Longrightarrow g = h \text{ for any two } g, h \in \mathcal{G}^\varepsilon$$

Moreover any (true)convex combination of two or more guarantees in $\mathcal{G}^\varepsilon$ stays in $\boldsymbol{G}^\varepsilon$ but leaves $\mathcal{G}^\varepsilon$.

Proof. *Suppose on the contrary that* $\mathcal{G}^-$ contains g, h and $\frac{1}{2}(g + h)$, all different. Fix $x_1 \in]L, H[$ and a contact profile $\widetilde{x}_{-1}$ of $\frac{1}{2}(g+h)$ at x_1. Clearly $\widetilde{x}_{-1}$ is also a contact profile of g and of h at x_1. Therefore by statement *ii)* in Lemma 7, almost surely in $x_1 \in]L, H[$ we have $\frac{dg}{dx_1}(x_1) = \frac{dh}{dx_1}(x_1) = \partial_1 \mathcal{W}(x_1, \widetilde{x}_{-1})$. We conclude that $g - h$ is a constant, and if it is not zero one of g, h is not maximal. The argument for larger convex combinations with general weights is entirely similar. ■

[3]Because we only require the Lipschitz property in each coordinate.

4 Sub- and super-modular functions $\mathcal{W}$

In this class of benefit and cost functions that includes most of our examples, the analysis of extremal guarantees greatly simplifies and this is key to the characterisation results of sections 5 and 6.

From now on until section 6 included, the type space $\mathcal{X}$ is a real interval $[L, H]$.

Definition 2 We call $\mathcal{W}$ *supermodular* if for all $i, j \in [n]$ and x, x' in $\mathcal{X}^{[n]}$ such that $x_k = x'_k$ for all $k \neq i, j$ we have

$$\{x_i \leq x'_i \text{ and } x_j \leq x'_j\} \Longrightarrow \mathcal{W}(x'_i, x_j; x_{-i,j}) + \mathcal{W}(x_i, x'_j; x_{-i,j}) \leq \mathcal{W}(x) + \mathcal{W}(x') \tag{11}$$

We say that $\mathcal{W}$ is strictly supermodular if whenever $(x_i, x_j) \ll (x'_i, x'_j)$ the RH inequality in (11) is strict. And $\mathcal{W}$ is submodular if the opposite inequality holds under the same premises.

Whenever the partial derivative $\partial_i \mathcal{W}(x)$ is defined in a neighborhood of x, super- (resp. sub-) modularity implies that it is weakly increasing (resp. decreasing) in x_j for all $j \neq i$. If $\partial_i \mathcal{W}(x)$ is strictly increasing (resp. decreasing) in x_j then $\mathcal{W}$ is strictly super- (resp. sub-) modular.

Whenever $\partial_i \mathcal{W}(x)$ is differentiable almost everywhere, the supermodularity property can be written as

$$\partial_{ij} \mathcal{W}(x) \geq 0 \text{ for all } i, j \in [n], i \neq j \text{ and a. e. in } x \in [L, H]^{[n]}$$

(and the opposite inequality for submodularity).

A well known consequence of super- (or sub-) modularity is this: if $(x_i, x_j) \ll (x'_i, x'_j)$ and the RH inequality in (*11*) is an equality, then in the interval $[(x_i, x_j), (x'_i, x'_j)]$ the function $(z_i, z_j) \rightarrow \mathcal{W}(z_i, z_j; x_{-i,j})$ is separably additive and its cross derivative $\partial_{ij} \mathcal{W}(z_i, z_j; x_{-i,j})$ is identically zero. We say that $\mathcal{W}$ is *locally i, j-additive* at the profile x if there is a rectangular neighborhood of (x_i, x_j) in which $\partial_{ij} \mathcal{W}(\cdot; x_{-i,j})$ is zero.

A strictly super- or sub-modular function like (in subsection 4.3 below) $\mathcal{W}(x) = F(\sum_i x_i)$ with F strictly convex or concave is not i, j-additive anywhere. But the function $\mathcal{W}(x) = \max_i \{x_i\}$ (section 2) is locally i, j-additive whenever $x_i \neq x_j$ (hence almost everywhere).

4.1 the unanimity guarantee is dominant on one side

This result explains the central role of the unanimity shares in the modular class of benefit and cost functions.

Notation: when the ordering of the coordinates does not matter $(z; \overset{k}{y})$ *represents the* $(k+1)$*-vector where one coordinate is* z *and* k *coordinates are* y*.*

Proposition 2 *If* $\mathcal{W}$ *is supermodular (resp. submodular), the unanimity utility (5) is the smallest upper (resp.largest lower) guarantee:* $\mathcal{G}^+ = \{una\}$ *(resp.* $\mathcal{G}^- = \{una\}$*).*

Proof We only do the proof for a supermodular $\mathcal{W}$; just exchange a few signs for a submodular one.

Step 1. We check $una \in \mathcal{G}^+$ first if $n = 2$. Pick two arbitrary types $x_1, x_2 \in [L, H]$, write $\delta = x_2 - x_1$ and consider the following function f and its derivative on $[0,1]$:

$$f(\lambda) = 2\mathcal{W}(x_1 + \lambda\delta, x_2) - (\mathcal{W}(x_1 + \lambda\delta, x_1 + \lambda\delta) + \mathcal{W}(x_2, x_2))$$

$$\frac{df}{d\lambda}(\lambda) = 2\delta(\partial_1 \mathcal{W}(x_1 + \lambda\delta, x_2) - \partial_1 \mathcal{W}(x_1 + \lambda\delta, x_1 + \lambda\delta))$$

Supermodularity implies that the term in parenthesis is zero or has the sign of $x_2 - (x_1 + \lambda\delta) = (1-\lambda)(x_2 - x_1)$; so its product with δ is non negative. This implies

$$f(1) = 0 \geq f(0) = 2\mathcal{W}(x_1, x_2) - (\mathcal{W}(x_1, x_1) + \mathcal{W}(x_2, x_2))$$

as desired.

Step 2 Induction on n*.* Assume statement $i)$ holds up to $(n-1)$ and fix an arbitrary n-person supermodular benefit $\mathcal{W}$ and profile x in $\mathcal{X}^{[n]}$. For any fixed i and x_i this implies that una is an upper guarantee of the $(n-1)$-benefit function $\mathcal{W}(\cdot; x_i)$:

$$\mathcal{W}(x) \leq \frac{1}{n-1} \sum_{j \neq i} \mathcal{W}(x_i; \overset{n-1}{x_j})$$

$$\Longrightarrow n\mathcal{W}(x) \leq \frac{1}{n-1} \sum_{(i,j) \in P} \mathcal{W}(x_i; \overset{n-1}{x_j}) \tag{12}$$

where P is the set of ordered pairs (i,j) in $[n]$.

Apply next the inductive assumption to the function $\mathcal{W}(\cdot; x_j)$ of $(n-1)$-variables at $(\overset{n-2}{x_j}; x_i)$:

$$\mathcal{W}(x_i; \overset{n-1}{x_j}) \leq \frac{1}{n-1}((n-2)\mathcal{W}(\overset{n}{x_j}) + \mathcal{W}(x_j; \overset{n-1}{x_j}))$$

Summing up both sides over all $(i,j) \in P$ and writing S for the summation in the RH of (12) gives

$$S \leq \frac{1}{n-1} S + (n-2) \sum_{j=1}^{n} \mathcal{W}(\overset{n}{x_j}) \Longrightarrow S \leq (n-1) \sum_{j=1}^{n} \mathcal{W}(\overset{n}{x_j})$$

Combining (12) with the latter inequality concludes the proof. ■

Lemma 8 *If $\mathcal{W}$ is strictly supermodular a maximal lower guarantee $g \in \mathcal{G}^-$ can have at most one unanimous contact point:*

$$\{g(\widetilde{x}_1) = una(\widetilde{x}_1) \text{ at some } \widetilde{x}_1\} \Longrightarrow \{g(x_1) < una(x_1) \text{ for all } x_1 \neq \widetilde{x}_1\}$$

The same is true if $\mathcal{W}$ is strictly submodular for minimal upper guarantees.

Proof Fix $\mathcal{W}, g \in \mathcal{G}^-$ as in the first statement and suppose g has two unanimous contact profiles $(\overset{n}{x_1})$ and $(\overset{n}{x_2})$ such that $x_1 < x_2$. Strict supermodularity implies, by repeated application of (11):

$$\mathcal{W}(x_2; \overset{n-1}{x_1}) - \mathcal{W}(\overset{n}{x_1}) < \mathcal{W}(\overset{n}{x_2}) - \mathcal{W}(x_1; \overset{n-1}{x_2})$$

$$\Longleftrightarrow \mathcal{W}(x_2; \overset{n-1}{x_1}) + \mathcal{W}(x_1; \overset{n-1}{x_2}) < \mathcal{W}(\overset{n}{x_1}) + \mathcal{W}(\overset{n}{x_2})$$

By our choice of x_1, x_2 the RH in the last inequality is written as

$$ng(x_1) + ng(x_2) = (g(x_2) + (n-1)g(x_1)) + (g(x_1) + (n-1)g(x_2)) \leq$$

$$\leq \mathcal{W}(x_2; \overset{n-1}{x_1}) + \mathcal{W}(x_1; \overset{n-1}{x_2})$$

and we have reached a contradiction. ■

Combining Lemmas 5 and 8 with Proposition 2 we see that if $\mathcal{W}$ is strictly supermodular:
the unanimity function is the unique minimal upper guarantee; there is an infinite set of maximal lower guarantees, each one with at most one unanimous contact point; and each unanimous profile is the contact point of at least one maximal lower guarantee. A symmetric statement holds for a strictly submodular function $\mathcal{W}$.

Extremal guarantees without any unanimous contact point are not a pathological occurrence: Proposition 4 in subsection 4.3 describes a clean family of such guarantees in the commons problem; more examples are in subsection 5.2.

4.2 two canonical incremental guarantees

All modular functions share two very simple and familiar extremal guarantees.

Proposition 3 *If $\mathcal{W}$ is supermodular (resp. submodular) the equations*

$$g_{inc}(x_i) = \mathcal{W}(x_i; \overset{n-1}{L}) - (n-1)una(L) \tag{13}$$

$$g^{inc}(x_i) = \mathcal{W}(x_i; \overset{n-1}{H}) - (n-1)una(H) \tag{14}$$

define two maximal lower (resp. minimal upper) guarantees in $\mathcal{G}^-$ (resp. $\mathcal{G}^+$) with unanimous contact points are at L and H respectively. We call them L-incremental and H-incremental.

Under g_{inc} the type L agent is the only one whose share is $una(L)$ irrespective of the distribution of other's types. If $\mathcal{W}$ is sub-modular and $\mathcal{W}(x)$ is a cost to share, this is good news because $una(L)$ is the unambiguous maximal lower bound of his cost share (Proposition 2). These interpretations flip twice: when we change cost to benefits and sub to supermodular.

An example is the cost function $\mathcal{W}_0(x) = \max_{i\in[n]}\{x_i\}$ in section 2 where $g_{inc} = g^+_{sa}$ and $g^{inc} = g^+_{ega}$ are the end points of $\mathcal{G}^+$ (Proposition 1): this confirms that g^+_{sa} favors type L and g^+_{ega} favors type H.

Actually the generic upper guarantee g^+_p also takes the incremental form starting from p:

$$g^+_p(x_i) = \frac{1}{n}p + (x_i - p)_+ = \mathcal{W}_0(x_i; \overset{n-1}{p}) - (n-1)una(p)$$

Theorem 1 in the next section 5 generalises this observation to a large class of benefit or cost functions.

Proof of Proposition 3 We give it for g_{inc} and $\mathcal{W}$ supermodular; it is identical for g_{inc} and/or $\mathcal{W}$ submodular by the usual inequality flips.

Check first that (13) defines a lower guarantee (4). If $n = 2$ this reduces to

$$\mathcal{W}(x_1, L) + \mathcal{W}(x_2, L) \leq \mathcal{W}(x_1, x_2) + \mathcal{W}(L, L)$$

a direct consequence of supermodularity (11).

Proceeding by induction on n, we assume that (13) defines a lower guarantee in problems with up to $n-1$ agents. Fix a supermodular function $\mathcal{W}$ and a profile x in $[L.H]^n$. We rearrange the LH of (4) for g_{inc} as

$$T = \sum_{i=1}^{n} \mathcal{W}(x_i; \overset{n-1}{L}) \leq \mathcal{W}(x) + (n-1)\mathcal{W}(\overset{n}{L}) \tag{15}$$

Because (13) define a lower guarantee of the 2-benefit function $\mathcal{W}(\cdot;\overset{n-2}{L})$ we have for all pairs i,j in $[n]$

$$\mathcal{W}(x_i,L;\overset{n-2}{L})+\mathcal{W}(x_j,L;\overset{n-2}{L})\leq\mathcal{W}(x_i,x_j;\overset{n-2}{L})+\mathcal{W}(\overset{n}{L})$$

Summing up these inequalities over the set Q of non ordered pairs in $[n]$ yields, after some rearranging

$$T\leq\frac{1}{n-1}\sum_{(i,j)\in Q}\mathcal{W}(x_i,x_j;\overset{n-2}{L})+\frac{n}{2}\mathcal{W}(\overset{n}{L}) \tag{16}$$

We fix agent i and write inequality (4), rearranged as (15), for the function g_{inc} of the $(n-1)$-benefit $\mathcal{W}(x_i;\cdot)$:

$$\sum_{j\neq i}\mathcal{W}(x_i;x_j,\overset{n-2}{L})\leq\mathcal{W}(x)+(n-2)\mathcal{W}(x_i;\overset{n-1}{L})$$

Summing up over all i gives

$$2\sum_{(i,j)\in Q}\mathcal{W}(x_i,x_j;\overset{n-2}{L})\leq n\mathcal{W}(x)+(n-2)T$$

Combining this last inequality and (16) gives

$$T\leq\frac{n}{2(n-1)}\mathcal{W}(x)+\frac{n-2}{2(n-1)}T+\frac{n}{2}\mathcal{W}(\overset{n}{L})$$

which, after some more rearranging is the desired inequality (15).

Step 4: g_{inc} is maximal. Definition (13) implies $g_{inc}(L)=una(L)$ and

$$g_{inc}(x_1)+(n-1)g_{inc}(L)=\mathcal{W}(x_1;\overset{n-1}{L})\text{ for all }x_1$$

and by Lemma 4 we are done. ■

4.3 the commons problem

In this much studied model (see section 1) an increasing and continuous function F transforms non negative inputs $x_i\in[L,H]$ by the n agents $(L\geq 0)$ into an output that they must share.

Example 1.A *Commons with complementary inputs*

Here $\mathcal{W}(x) = F(\min_i x_i)$ so the inputs are perfect complements. The function $\mathcal{W}$ is supermodular, and the problem is actually isomorphic to the canonical one in section 2 by changing the type variable $y_i = F(x_i)$, inverting signs to turn max into a min and checking that the solutions of (4) commute with these changes. So $una(x_i) = \frac{1}{n}F(x_i)$ is the smallest upper guarantee and the maximal lower guarantees are parametrised by $p \in [L, H]$ as

$$g_p^-(x_i) = \frac{1}{n}F(p) + \min\{F(x_i) - F(p), 0\} \text{ for } x_1 \in [L, H]$$

Any input (effort) x_i weakly above the bechmark p guarantees at least the share $\frac{1}{n}F(p)$ and at most $\frac{1}{n}F(x_i)$. A "slacker" inputing $x_i < p$ can get as low as $F(x_i) - \frac{n-1}{n}F(p)$ which can be negative. For instance $g_p^-(0) = -\frac{n-1}{n}F(p)$ if $L = 0$ and $F(0) = 0$: no output is produced if $x_1 = 0$ and agent 1, perhaps helped by other slackers s. t. $g_p^-(x_i) < 0$, if any, pays in cash $\frac{1}{n}F(p)$ to each "responsible" agent whose (wasted) effort reaches the benchmark.[4]

It is much harder to solve system (4) in the more familiar case where the inputs are perfect substitutes: $\mathcal{W}(x) = F(\sum_i x_i)$. Recall the dual interpretation of this case where x_i is agent i's demand of output and y_i the share of the input she must contribute.

In addition to the two incremental guarantees in Proposition 3, we discover $n - 2$ other extremal guarantees of the same incremental type $g(x_1) = \mathcal{W}(x_1, c_{-1}) - \gamma$ (where c_{-1} and γ are constant).

Proposition 4 *Commons with substitutable inputs*
i) If F is concave (resp. convex) and increasing on $[L, H] \subset \mathbb{R}_+$ the function $x \to \mathcal{W}(x) = F(\sum_i x_i)$ is submodular (resp. supermodular) and admits the following minimal upper (resp. maximal lower) guarantees $g_{\ell,h}$, where ℓ, h are two non negative integers s. t. $\ell + h = n - 1$. For all $x_i \in [L, H]$

$$g_{\ell,h}(x_i) = F(x_i + (\ell L + hH)) - \{\frac{\ell}{n}F((\ell+1)L + hH) + \frac{h}{n}F(\ell L + (h+1)H)\} \tag{17}$$

Here $g_{n-1,0} = g_{inc}$ and $g_{0,n-1} = g^{inc}$ are the two guarantees identified in Proposition 3.
ii) If F is strictly concave (resp. convex) only g_{inc} and g^{inc} have an unanimous contact point.

Proof For easier reading we set $Z = \ell L + hH$.
Statement i) The concavity of F implies at once that $\mathcal{W}$ is submodular.

[4] Of course this matches the interpretation of Proposition 1 for assigning a bad in section 2.

We check first the feasibility inequality (4) which we rewrite as

$$\sum_i F(x_i + Z) \geq F(x_N) + \ell F(Z + L) + hF(Z + H) \tag{18}$$

The partial derivatives (defined a. e.) of the function $\pi(x) = \sum_i F(x_i + Z) - F(x_N)$ are $\frac{\partial \pi}{\partial x_i}(x) = \frac{dF}{dx}(x_i + Z) - \frac{dF}{dx}(\sum_i x_i)$ therefore its gradient $\Delta\pi(x)$ is such that we have for all $x, x' \in [L, H]^{[n]}$

$$< \Delta\pi(x) - \Delta\pi(x'), x - x' >= \sum_i (\frac{dF}{dx}(x_i + Z) - \frac{dF}{dx}(x'_i + Z)).(x_i - x'_i) \leq 0$$

because $\frac{dF}{dx}$ decreases weakly. Therefore π is concave, moreover $\Delta\pi(x) = 0$ at $x = (\overset{n}{\frac{1}{n-1}Z})$ where π reaches its minimum. So (18) reduces to

$$(n-1)F(\frac{n}{n-1}Z) \geq \ell F(Z + L) + hF(Z + H)$$

which is just the concavity inequality for F.

To check that $g_{\ell,h}$ is maximal we fix x_1 and invoke Lemma 4 at the profile $(x_1, \overset{\ell}{L}, \overset{h}{H})$ where we compute

$$g_{\ell,h}(x_1) + \ell g_{\ell,h}(L) + hg_{\ell,h}(H) = F(x_1 + \ell L + hH)$$

Statement ii) If $g_{\ell,h}$ has a unanimous contact point at x_1 we have

$$nF(x_1 + Z) = F(nx_1) + \ell F(Z + L) + hF(Z + H)$$

If F is strictly concave and ℓ, h are both positive this contradicts the concavity inequality. If ℓ or h is zero we find g_{inc} *and* g^{inc} with unanimous contact points at L and H respectively. ∎

Example 6 in subsubsection 7.1 describes another family of extremal guarantees in the commons problem: it is infinite of dimension 1 and, like the discrete family just described, it connects the two canonical incremental guarantees; all elements of the family have a unanimous contact point.

Example 1.B: *Commons with multiplicative inputs*

Each agent i contributes an effort input x_i in $[L, H]$ and total output is the product of individual efforts

$$\mathcal{W}(x) = x_1 x_2 \cdots x_n \text{ for } x \in [L, H]^{[n]}$$

For instance $[L, H] \subset [0,1]$, the effort x_i is the probability that agent i "succeeds" (independent of other agents' efforts) and one unit of output is produced if and only if everyone succeeds.

The simple change of variable $y_i = \ln(x_i)$ and of function

$$\widetilde{\mathcal{W}}(y) = \exp(\sum_i y_i)$$

shows that this problem is isomorphic to the commons with substitutable inputs and convex production function $F = \exp$. Both $\mathcal{W}$ and $\widetilde{\mathcal{W}}$ are supermodular and the maximal lower guarantee $\widetilde{g}_{\ell,h}$ described in Proposition 4 for $\widetilde{\mathcal{W}}$ corresponds to $\widehat{g}_{\ell,h}$ for $\mathcal{W}$:

$$\widehat{g}_{\ell,h}(x_i) = L^\ell H^h (x_i - \frac{1}{n}(\ell L + hH))$$

The two incremental guarantees in Proposition 3, for ℓ or $h = n-1$ are

$$\widehat{g}_{inc}(x_i) = L^{n-1}(x_i - \frac{n-1}{n}L) \text{ and } \widehat{g}^{inc}(x_i) = H^{n-1}(x - \frac{n-1}{n}H)$$

Their graphs are tangent to the unanimity function $una(x_i) = \frac{1}{n}x_i^n$ at L and H respectively.[5]

If $L > 0$ then $\widehat{g}_{inc}(x_i)$ is strictly positive for all x_i: providing a minimal effort L is not punished, it still guarantees the share $una(L)$. On the contrary $\widehat{g}^{inc}$ rewards high effort and guarantees $una(H)$ for the effort level H: this is feasible by charging penalties to the "slackers" who input less than $\frac{n-1}{n}H$.[6]

The $n-2$ other lower guarantees $\widehat{g}_{\ell,h}$ allow the manager to adjust the critical effort level $\frac{\ell L + hH}{n}$ guaranteeing a positive benefit share along a n-grid from L to $\frac{n-1}{n}H$.

5 Rank-separable functions and general incremental guarantees

We solve completely the functional inequality (4) for a large class of functions $\mathcal{W}$ in which $\mathcal{W}_0(x) = \max_i x_i$ (section 2) is the simplest element.

[5]Whereas with substitutable inputs their graphs are translations of that of F.

[6]For instance if all agents but 1 input effort H while agent 1 only provides effort L, the latter must pay $|g^{sa}(L)| = H^{n-1}(\frac{n-1}{n}H - L)$; if $L = 0$ this agent must pay $una(H)$ to every agent inputing the effort H!

The function $\mathcal{W}_0$ is submodular but not strictly so. For any distinct $i, j \in [n]$ and $x \in \mathcal{X}$ it satisfies

$$\text{for all } i, j \in [n] \text{ and } x \in [L, H]^{[n]} : x_i \neq x_j \Longrightarrow \partial_{ij}\mathcal{W}(x) = 0 \tag{19}$$

This is the defining property of the class in question. Applying (19) in the convex open subset of $[L, H]^{[n]}$ defined by the strict inequalities $x_1 < x_2 < \cdots < x_n$ we see that $\mathcal{W}$ must be separably additive. Because $\mathcal{W}$ is symmetric in its variables this defines $\mathcal{W}(x)$ whenever all coordinates of x are different, and finally everywhere as $\mathcal{W}$ is continuous.

Notation: $(x^1, x^2, \cdots, x^n)$ *is the decreasing order statistics of* $x = (x_1, \cdots, x_n)$ (so $x^1 = \max_i x_i$ and $x^n = \min_i x_i$).

Definition 3 *The function* $\mathcal{W}$ *on* $[L, H]^{[n]}$ *is called rank-separable if there exist* n *weakly increasing, equicontinuous real valued functions* w_k *on* $[L, H]$ *s. t.* $w_k(L) = w_\ell(L)$ *for all* $k, \ell \in [n]$ and

$$\mathcal{W}(x) = \sum_{k=1}^{n} w_k(x^k) \text{ for all } x \in [L, H]^{[n]} \tag{20}$$

Lemma 9 *The function (20) is supermodular if and only if we have*

$$\text{for all } k \in [n]\text{: } \frac{dw_k}{dx}(x_i) \leq \frac{dw_{k+1}}{dx}(x_i) \text{ a. e. in } x_i \in [L, H] \tag{21}$$

*and is submodular **iff** the opposite inequalities hold.*

Proof Note first that "x_i is of rank k in a certain profile" only makes sense if x_i is different from any other coordinate.

Fix $\mathcal{W}$ defined by (20): it is equicontinuous and weakly increasing. For "only if" we assume that $\mathcal{W}$ is supermodular.Fix two agents i, j and a $(n-2)$-profile $x_{-ij} \in [L, H]^{[n] \diagdown i,j}$. For any 4-tuple x_i, y_i, x_j, y_j such that $x_i > y_i$ and $x_j > y_j$ supermodularity means

$$\mathcal{W}(x_i, x_j; x_{-ij}) - \mathcal{W}(y_i, x_j; x_{-ij}) \geq \mathcal{W}(x_i, y_j; x_{-ij}) - \mathcal{W}(y_i, y_j; x_{-ij}) \tag{22}$$

Suppose $L < y_i < x_i < H$ and pick an arbitrary rank $k, k \leq n-1$: we can choose x_{-ij}, x_j and y_j s. t. in the profiles on the RH x_i and y_i are of rank k, while after increasing y_j to x_j they are of rank $k+1$ in the profiles on the LH. Then the inequality (22) reads

$$w_{k+1}(x_i) - w_{k+1}(y_i) \geq w_k(x_i) - w_k(y_i)$$

As x_i, y_i can be chosen arbitrary close to each other, this proves (21) at any interior point of $[L, H]$ where w_k is differentiable (that is, a. e.).

For "if" we assume (21) and fix x_{-ij}. For any x_i, y_j s. t. x_i has rank k in $(x_i, y_j; x_{-ij})$ we have $\partial_i \mathcal{W}(x_i, y_j; x_{-ij}) = \frac{dw_k}{dx}(x_i)$ (a. e.): if y_j is below x_i and jumps up to x_j above x_i then by (21) $\partial_i \mathcal{W}(x_i, x_j; x_{-ij})$ also increases (weakly) to $\frac{dw_{k+1}}{dx}(x_i)$. If x_i is not isolated in the profile $(x_i, y_j; x_{-ij})$ the same argument applies to the left and right derivatives of $\mathcal{W}$ in x_i. ■

Our first main result is that all extremal guarantees of any modular rank-separable function are of this same simple form.

Theorem 1 *Fix a rank-separable and super-(resp. sub-)modular function $\mathcal{W}$ as in Lemma 9.*
i) Each choice of $c = (c_1, \cdots, c_{n-1}) \in [L, H]^{n-1}$ defines a *maximal lower (resp. minimal upper) guarantee as follows:*

$$g_c(x_i) = \mathcal{W}(x_i; c) - \frac{1}{n}\sum_{k=1}^{n-1} \mathcal{W}(c_k; c) \text{ for all } x_i \in [L, H] \tag{23}$$

ii) Conversely every maximal lower (resp. minimal upper) guarantee of W takes this form.

Proof in the Appendix.

Note that if the parameter $c = (\overset{n-1}{c_0})$ is unanimous the corresponding extremal guarantee is

$$g_{c_0}(x_i) = \mathcal{W}(x_i; \overset{n-1}{c_0}) - \frac{n-1}{n}\mathcal{W}(\overset{n}{c_0})$$

precisely similar to the two canonical incremental guarantees, with a unanimous contact profile $(\overset{n}{c_0})$. Just after the statement of Proposition 3 we noticed that this captures all the minimal upper guarantees of the canonical $\mathcal{W}_0(x) = \max_i x_i$.

Example 2.A *Cost sharing of a capacity (revisited)*

In this more refined version of the cost sharing story in section 2 we take into account congestion costs. Given a profile of demands x the first x^n units of capacity are used by everyone so their per unit cost γ_n is higher than γ_{n-1}, that of the next $(x^{n-1} - x^n)$ units used by only $n-1$ agents, and so on. With the conventions $x^{n+1} = \gamma_0 = 0$ we define $w_k = \gamma_k - \gamma_{k-1}$ and the cost function

$$\mathcal{W}(x) = \sum_{k=1}^{n} \gamma_k (x^k - x^{k+1}) = \sum_{k=1}^{n} w_k x^k \text{ for } x \in [L, H]^{[n]}$$

By Lemma 9 $\mathcal{W}$ is super- (resp. sub-) modular iff $\gamma_k - \gamma_{k-1}$ decreases weakly in k, i. e., the marginal cost of congestion increases (resp. decreases) weakly with the congestion. Both properties are plausible in different contexts, and Theorem 1 describes the sets $\mathcal{G}^+$ and $\mathcal{G}^-$ in both cases.

With two agents and concave congestion costs $\mathcal{W}(x) = w_1 x^1 + w_2 x^2$ with $w^1 > w^2 > 0$. The set $\mathcal{G}^+$ is parametrised by $c \in [L, H]$ as follows:

$$g_c(x_i) = w_2 x_i + \frac{1}{2}(w_1 - w_2)c \text{ for } x_i \in [L, c] \; ; \; = w_1 x_i - \frac{1}{2}(w_1 - w_2)c \text{ for } x_i \in [c, H]$$

$$g_c(x_i) \geq \frac{1}{2}(w_1 + w_2)x_i = una(x_i) \text{ for } x_i \in [L, H]$$

The manager looking to minimise the maximal gap $\max_{0 \leq x_i \leq H}\{g_c(x_i) - una(x_i)\}$ will choose $c = \frac{1}{2}(H + L)$. If instead they minimise the expected gap for a given distribution of types, the computations remain simple. With three or more agents the corresponding optimisations problems are still linear programs.

Example 2.B *Sharing the cost of being connected ([24])*

After each agent i chooses a location x_i in the interval $[L, H]$ the manager must cover the cost of connecting them (e g by building a road) which we assume linear in the largest distance between agents:

$$\mathcal{W}(x) = x^1 - x^n \text{ for } x \in [L, H]^{[n]}$$

The high level question is whether one should be penalised for being far away at the periphery of the distribution of agents, and if so, by how much. Once again the extremal guarantees go a long way toward answering this question, while still leaving much room to choose precise shares between those bounds.

The cost is submodular and the largest lower guarantee is $una(x_i) \equiv 0$: everyone's best case is to pay nothing (if they happen to be on the same spot). By Theorem 1 a minimal upper guarantee involves the choice of $n-1$ variables c_k but it is easy to check in equation (23) that for any $n \geq 3$ only the largest and smallest values c^+ and c^- matter:

$$g_c(x_i) = (\max\{x_i, c^+\} - \min\{x_i, c^-\}) - \frac{n-1}{n}(c^+ - c^-) \text{ for } x_i \in [L, H]$$

where $c = (c^+, c^-)$. Writing $\mu = \frac{1}{n}(c^+ - c^-)$ we develop g_c as follows

$$g_c(x_i) = \mu \text{ if } c^- \leq x_i \leq c^+$$

$$g_c(x_i) = \mu + (c^- - x) \text{ if } L \leq x_i \leq c^-; \; g_c(x_i) = \mu + (x - c^+) \text{ if } c^+ \leq x_i \leq H$$

So an agent with type in the benchmark interval $[c^-, c^+]$ has the smallest possible worst cost share μ, and one outside this interval could pay, in addition to μ, the full connecting cost to the benchmark.

If $c^- = c^+ = c^*$ an agent locating at c^* pays nothing (irrespective of other agents' location) and $g_c(x_i) = |x - c^*|$. While if $[c^-, c^+] = [L, H]$ the worst cost share is $\frac{1}{n}(H - L)$ for everybody. These resemble the stand alone and egalitarian upper guarantees of section 2.

Remark 2 Sharing the traveling costs to the median. Assume the facility is located at the median of the agents to minimise the transportation costs that they must share. If $n = 2q + 1$ *total costs are* $\mathcal{W}(x) = \sum_{k=1}^{q} x^k - \sum_{\ell=1}^{q} x^{\ell+q+1}$, again submodular by Lemma 9 and again $una(x_i) \equiv 0$. *But now the choice of a rank-specific minimal upper guarantee in Theorem 1 involves* $2q$ *parameters and allows a much more nuanced evolution of shares.*

Example 2.C *Ranked commons*

Fix a rank $k \in [n]$. Each agent i produces an individual effort x_i; to achieve the output level $y = F(z)$ we need at least k agents contributing an effort at least z:

$$\mathcal{W}_k(x) = F(x^k) \text{ for } x \in [L, H]^{n]} \tag{24}$$

where F is continuous and increasing.

This function is neither sub- nor super-modular (even if F is linear) and $una(x_i) = \frac{1}{n}F(x_i)$ is not a lower guarantee unless $k = 1$, and not an upper guarantee unless $k = n$. It is nevertheless easy to describe the sets $\mathcal{G}_k^{\pm}$. The proof given in the appendix mimicks that of Proposition 1.

For $k \leq n - 1$ the set $\mathcal{G}_k^+$ is parametrised by $p \in [L, H]$ as follows:

$$g_{k,p}^+(x_i) = \frac{1}{n}F(p) + \frac{1}{k}(F(x_i) - F(p))_+ \text{ for } x_i \in [L, H]$$

and $\mathcal{G}_n^+ = \{una\}$.

Similarly for $k \geq 2$ the set $\mathcal{G}_k^-$ is parametrised by $q \in [L, H]$ as:

$$g_{k,q}^-(x_i) = \frac{1}{n}F(q) + \frac{1}{n - k + 1}(F(x_i) - F(q))_-$$

and $\mathcal{G}_1^- = \{una\}$.

Unlike in all our other examples (starting with Proposition 1) there is here a choice for both upper and lower bounds.

If $p = q$ this benchmark effort guarantees the share $\frac{1}{n}F(p) = una(p)$; if $p \neq q$ we have $g_{k,p}^+(x_i) < g_{k,q}^-(x_i)$ for all x_i. See Figure X.

6 Two agents: general characterisation

We fix $\mathcal{X} = [L, H]$ and a supermodular function $\mathcal{W}(x_1, x_2)$ on $\mathcal{X}^2$. If $g \in \mathcal{G}^-$ is a maximal lower guarantee its contact correspondence φ

$$\varphi(x_1) = \{x_2 \in [L, H] | g(x_1) + g(x_2) = \mathcal{W}(x_1, x_2)\} \tag{25}$$

is non empty at each $x \in [L, H]$ (Lemma 4). Its graph is written $\Gamma(\varphi)$.

Lemma 10 *If $\Gamma(\varphi)$ contains (x_1, x_2) and (x'_1, x'_2) s.t. $(x_1, x_2) \ll (x'_1, x'_2)$, then $(x_1, x'_2), (x'_1, x_2) \in \Gamma(\varphi)$ as well, and W is not strictly supermodular.*

Proof We sum up the two equalities in (25) for (x_1, x_2) and $(x_1\prime, x_2\prime)$:

$$\mathcal{W}(x_1, x_2) + \mathcal{W}(x'_1, x'_2) = \{g(x_1) + g(x'_2)\} + \{g(x'_1) + g(x_2)\} \leq \mathcal{W}(x_1, x'_2) + \mathcal{W}(x'_1, x_2)$$

Combined with the supermodular inequality (11) this gives an equality. ■

Lemma 11 *Fix $n = 2$ and a strictly super- (resp. sub-)modular benefit function $\mathcal{W}$ (Definition 2 section 4). Then for any $g \in \mathcal{G}^-$ (resp. $\mathcal{G}^+$) with contact correspondence φ we have:*
i) $\Gamma(\varphi)$ is symmetric: $x_2 \in \varphi(x_1) \Longleftrightarrow x_1 \in \varphi(x_2)$ for all x_1, x_2.
ii) φ is convex valued: $\varphi(x_1) = [\varphi^-(x_1), \varphi^+(x_1)]$ for all x_1, single-valued a.e., and upper-hemi-continuous (its graph is closed).
iii) φ^- and φ^+ are weakly decreasing and $x_1 \leq x'_1 \Longrightarrow \varphi^-(x_1) \geq \varphi^+(x'_1)$; φ is the u.h.c. closure of both φ^- and *φ^+.*
iv) $\varphi(L)$ contains H and $\varphi(H)$ contains L.
v) φ has a unique fixed point a: $a \in \varphi(a)$, and a is an end-point of $\varphi(a)$.

Proof in the Appendix.

Theorem 2 *Fix a strictly super- (resp. sub-) modular function $\mathcal{W}$, continuously differentiable in $[L, H]^2$.*
i) For any correspondence φ as in Lemma 9, the following equation

$$g(x_1) = \int_a^{x_1} \partial_1 \mathcal{W}(t, \varphi(t))dt + una(a) \tag{26}$$

defines a maximal lower guarantee $g \in \mathcal{G}^-$ (resp. $\mathcal{G}^+$).
ii) Conversely if g is a guarantee in $\mathcal{G}^-$ (resp. $\mathcal{G}^+$), its contact correspondence φ ((25)) is as in Lemma 9 and g takes the form (26).

Proof in the Appendix.

Note that the integral expression (*26*) of the extremal guarantee is the same for sub- and super-modular functions $\mathcal{W}$.

Theorem 2 shows that the sets $\mathcal{G}^{\pm}$ are very large even if $\mathcal{X}$ is one-dimensional. After choosing the benchmark type a which guarantees the share $una(a)$, we can pick any decreasing single-valued function $\overline{\varphi}$ from $[L,a]$ into $[a,H]$ mapping L to H, then fill the (countably many) jumps down to create the correspondence φ of which the graph connects (L,H) to (a,a), and finally extend φ to $[a,H]$ where its graph is the symmetric of its graph in $[L,a]$ around the diagonal of $[L,H]^2$ so that φ also maps H to L.

To add Figure.

Example 3 *Commons with substitutable inputs (continued from Proposition 4)*

This is the model subsection 4.3 with only two agents: $\mathcal{W}(x) = F(x_1+x_2)$ and F is increasing and strictly concave (or strictly convex) on $[L,H]$.

We illustrate the minimal upper (or maximal lower) guarantees defined by (*26*) for simple choices of the function φ.

Although Proposition 4 when $n=2$ delivers only the two canonical incremental guarantees (Proposition 3), there are two one-dimensional families of "two-piece-incremental" guarantees, parametrised by the fixed point a of φ.

In the first family $\varphi(\cdot) \equiv H$ on $[L,a]$ then $\varphi(\cdot) \equiv a$ on $[a,H]$ (with appropriate jumps down at a and H (Lemma 11). Equation (*26*) gives g_a in $\mathcal{G}^{\pm}$:

$$g_a(x_1) = F(x_1+H) - F(a+H) + \frac{1}{2}F(2a) \text{ for } x \in [L,a]$$

$$g_a(x_1) = F(x_1+a) - \frac{1}{2}F(2a) \text{ for } x \in [a,H]$$

In the second family φ equals first a on $[L,a]$ then L on $[a,H]$:

$$g_a(x_1) = F(x_1+a) - \frac{1}{2}F(2a) \text{ for } x \in [L,a]$$

$$g_a(x_1) = F(x_1+L) - F(a+L) + \frac{1}{2}F(2a) \text{ for } x \in [a,H]$$

A very different type of guarantee obtains from (*26*) and the function $\varphi(x_1) = L+H-x_1$, of which the graph is the anti-diagonal of $[L,H]^2$:

$$g^*(x_1) = \frac{1}{2}F(L+H) + \frac{dF}{dx}(L+H)(x_1 - \frac{L+H}{2}) \text{ for } x \in [L,H]$$

This is simply the tangent at $\frac{L+H}{2}$ to the unanimity function $una(x_1) = \frac{1}{2}F(2x_1)$. In Example 4, subsection 7.1 we describe yet another one-dimensional family of guarantees combining features of both the incremental-like and the tangent guarantee above.

7 Two open questions

7.1 a characterisation result for $n \geq 3$?

The key to Theorem 2 for two agent problems is the deep understanding of the contact correspondence $\{(x_1, \varphi(x_1)); x_1 \in \mathcal{X}\}$ of any extremal guarantee (Lemmas 10, 11). We could not gain a similar understanding of this correspondence with three or more agents, where it takes the form $\{(x_1, \varphi_1(x_1), \cdots, \varphi_{n-1}(x_1)); x_1 \in \mathcal{X}\}$.

To emulate the proof technique of Theorem 1 in an n-person problem we now assume that each type x_1 has at least one diagonal contact profile where all $\varphi_k(x_1)$ coincide. This is a considerable simplification: it rules out $n-2$ of the extremal guarantees we described in the commons problem (Proposition 4) and most of them for rank-separable functions (Theorem 1).

Equation (7) becomes

$$\forall x_i \in \mathcal{X}\ \exists x_j \in \mathcal{X} : g(x_i) + (n-1)g(x_j) = \mathcal{W}(x_i; \overset{n-1}{x_j}) \tag{27}$$

As in (25) the set of its solutions x_j defines a correspondence $x_i \to \theta(x_i)$ from $[L, H]$ into itself. It is easy to check that most properties of φ listed in Lemmas 10 and 11 hold true for θ: it is convex valued, weakly decreasing and upper-hemi-continuous, with a unique fixed point a . However, it is not symmetric and its range can be smaller than $[L, H]$.

Exactly like in the proof of statement ii) of Theorem 2 we see that any extremal guarantee in the relevant $\mathcal{G}^\varepsilon$ where the contact correspondence contains a diagonal selection θ must take the form

$$g(x_i) = \int_a^{x_i} \partial_1 \mathcal{W}(t, \overset{n-1}{\theta(t)})dt + una(a) \tag{28}$$

The contact equation (27) is now

$$\int_a^{x_i} \partial_1 \mathcal{W}(t, \overset{n-1}{\theta(t)})dt + (n-1)\int_a^{\theta(x_i)} \partial_1 \mathcal{W}(t, \overset{n-1}{\theta(t)})dt = \mathcal{W}(x_i, \overset{n-1}{\theta(x_i)}) - \mathcal{W}(\overset{n}{a}) \tag{29}$$

but unlike in the two person case this does not automatically follow, except for $x_i = a$. Taking derivatives on both sides gives, after simple computations, a functional equation in θ:

$$\text{for all } x_i \in [L, H] : \frac{d\theta}{dx}(x_i) = 0 \text{ and/or } \partial_1 \mathcal{W}(\theta(x_i), \overbrace{\theta \circ \theta(x_i)}^{n-1}) = \partial_2 \mathcal{W}(x_i, \overset{n-1}{\theta(x_i)})$$

In addition we still need to check that g defined by (28) meets the relevant side of inequality (4). Though hopeless in the general case, this approach allows us to identify some new solutions of the commons problem.

Example 4 *Commons with substitutable inputs (continued)*

We assume as in subsection 4.3 that F strictly concave and increasing. For simplicity of the equations its domain is $[0, H]$ and $F(0) = 0$. In Proposition 4 we identified n extremal guarantees of the incremental type for this problem; below we uncover another subset $\{g_a; a \in [0, H]\}$ of $\mathcal{G}^+$, parametrised by the fixed point of g_a and built around the tangents supporting the unanimity function.[7] It is doubtful that these two very different subsets of $\mathcal{G}^+$ capture the entire set $\mathcal{G}^+$ of this most studied version of the commons problem.

Definition (28) and the functional equation (29) are now

$$g(x_i) = \int_a^{x_i} \frac{dF}{dx}(t + (n-1)\theta(t))dt + \frac{1}{n}F(na)$$

$$\text{for all } x_i \in [0, H] : \frac{d\theta}{dx}(x_i) = 0 \text{ and/or } \theta \circ \theta(x_i) = \frac{n-2}{n-1}\theta(x_i) + \frac{1}{n-1}x_i \tag{30}$$

The identity $\frac{d\theta}{dx} \equiv 0$ gives $\theta(\cdot) \equiv a$ and the incremental equation at a

$$g(x_i) = F(x_i + (n-1)a) - \frac{n-1}{n}F(na) \text{ for all } x_i$$

but g violates the inequality (4) if $a \in]0, H[$, for instance at $x = (a + \varepsilon, a - \varepsilon, \overset{n-2}{a})$ because F is strictly concave.

The affine solutions of the functional equation RH of (30) take the form

$$\theta(x_i) = a + \frac{1}{n-1}(a - x_i)$$

which maps $[0, H]$ into itself if and only if $a \in [\frac{1}{n}H, \frac{n-1}{n}H]$. In this case we find the following guarantees:

$$g_a^-(x_i) = \frac{1}{n}F(na) + \frac{dF}{dx}(na)(x_i - a) \text{ for } x_i \in [0, H] \tag{31}$$

The graph of g_a^- is tangent to that of the unanimity guarantee at a; and the RH of inequality (4) follows at once from the tangent inequality of F at na.

[7] Naturally if F strictly convex, increasing, and $F(0) = 0$, the same functions g_a describe maximal lower guarantees.

If $a \in [0, \frac{1}{n}H]$ we find a solution θ of (30) meeting the RH for $x_i \in [0, na]$ and the LH for $x_i \in [na, H]$. So $g_a(x_i)$ is given by (31) in $[0, na]$, and in $[na, H]$ by

$$g_a^-(x_i) = F(x_i) - \frac{n-1}{n}F(na) + (n-1)\frac{dF}{dx}(na)a$$

Similarly if $a \in [\frac{n-1}{n}H, H]$ then $g_a^-(x_i)$ is as in (31) for $x_i \in [na - (n-1)H, H]$ and

$$g_a^-(x_i) = F(x+(n-1)H) - \frac{n-1}{n}F(na) - (n-1)\frac{dF}{dx}(na)(H-a) \text{ for } x_i \in [0, na-(n-1)H]$$

Figure X illustrates the three types of guarantees just described.

7.2 multi-dimensional types and a decentralisation question

The general results in section 3 apply to functions $x \to \mathcal{W}(x)$ of any number m of real variables x_i. To test their power for $m \geq 2$ a natural first step is the following simple generalisation of the problem in section 2.

We must allocate a set $A = \{a, b, \cdots\}$ of indivisible items between agents with utilities additive across items: efficiency requires to assign each item to a corresponding efficient agent. Agent i's type is $x_i = (x_{ia})_{a \in A} \in [L, H] \subset \mathbb{R}^A$ and the total benefit

$$\mathcal{W}_A(x) = \sum_{a \in A} \max_{i \in [n]} \{x_{ia}\} \tag{32}$$

is additively separable in the m items. This suggest that the extremal guarantees of $\mathcal{W}_A$ are additively separable as well.

Proposition 5

i) The unanimity function $una(x_i) = \frac{1}{n}\sum_{a \in A} x_{ia}$ is the largest lower guarantee of $\mathcal{W}_A$.

ii) Pick for each a a minimal upper guarantee $g_{p_a}^+$ of $\mathcal{W}_a(x) = \max_{1 \leq i \leq n}\{x_{ia}\}$ with parameter $p_a \in [L_a, H_a]$ (Proposition 1). Then

$$g^+ = \sum_{a \in A} g_{p_a}^+$$

is a minimal upper guarantee of $\mathcal{W}_A$.

iii) For $n = 2$ every minimal upper guarantee of $\mathcal{W}_A$ takes the above form for some choice of $p_a \in [L_a, H_a]$ and *$p_b \in [L_b, H_b]$.*

Statement i) holds because $\mathcal{W}_A$ is submodular. Checking statement ii) is straightforward.

The proof of the equally intuitive statement iii) takes (much) more work, it is done in the Appendix. A very plausible conjecture is that it holds for any number m of items.

We can ask a more general decentralisation question. Suppose each type has two components $x_i = (x_i^1, x_i^2) \in \mathcal{X}^1 \times \mathcal{X}^2 = \mathcal{X}$ that the function $\mathcal{W}$ separates:

$$\mathcal{W}(x) = \mathcal{W}_1(x^1) + \mathcal{W}_2(x^2) \text{ for all } x \in \mathcal{X}$$

Statement ii) in Proposition 5 still holds, and so does statement i) (or its dual) if $\mathcal{W}$ is sub- (or super-) modular.

We conjecture that the converse separability property iii) applies to more families of component functions than $\mathcal{W}(x) = \max_{i\in[n]}\{x_i\}$, for instance to the rank separable ones. We have not been able to prove or find a counter example of this property for general modular functions $\mathcal{W}$.

8 Two take-home points

1) Given the evaluation of the benefits or costs of resources held in the common property regime, choosing a pair of extremal guarantees (one lower, one upper) is an endogenous and story-free way to interpret individual rights. It severely restricts the range of feasible allocations in ways which, in many simple examples, have clear normative meaning.

2) For sub- or super-modular functions $\mathcal{W}$ a typical choice above is on one side an infinite set (perhaps of infinite dimension), and the single unanimity guarantee on the other. The prominent role of the latter in the modular class confirms its importance in other contexts such as the division of private commodities or the exploitation of a common production function.

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9 Appendix: missing proofs

9.1 Theorem 1

We fix $\mathcal{W}$ given by (20) and assume that it is submodular, so that $\frac{dw_k}{dx}(\cdot)$ decreases weakly with k.

Statement i). We must show that g_c defined by (23) is in G^+. Because $g_c(x_i)$ and $\mathcal{W}(x_i;c)$ are continuous in x_i, c it is enough to prove the feasibility inequality (4) for strictly decreasing sequences $\{x_\ell\}_1^n$ and $\{c_k\}_1^{n-1}$ such that $H > c_1$ and $c_{n-1} > L$ and moreover $x_\ell \neq c_k$ for all ℓ, k.

Step 1 Call the profile of types x^* *regular* if

$$x_1^* > c_1 > x_2^* > c_2 > \cdots > c_{k-1} > x_k^* > c_k > \cdots > c_{n-1} > x_n^* \tag{33}$$

then compute

$$\sum_1^n g_c(x_k^*) = \sum_1^n \mathcal{W}(x_k^*, c) - \sum_1^{n-1} \mathcal{W}(c_k, c) = \sum_1^{n-1} (w_k(x_k^*) - w_k(c_k)) + \mathcal{W}(x_n^*, c) = \mathcal{W}(x^*)$$

so that x^* is a contact profile of g_c.

Step 2 For any two (strictly decreasing) n-sequences x, x' we say that x' is reached from x by an *elementary jump up above* c_k if $x_{-k} = x'_{-k}$; c_k is adjacent to some x_ℓ in the ordered sequence combining the x_i-s and c_j-s; $x_\ell < c_k < x'_\ell$ and c_k and x'_ℓ are still adjacent after the jump. The definition of an elementary jump down below c_k is exactly symmetrical.

In this step we claim that for any $\widetilde{x}$ we can find a regular profile x^* and a sequence $\sigma = \{x^* = x^1, \cdots, x^t, \cdots, x^T = \widetilde{x}\}$ such that

1) each step from x^t to x^{t+1} is an elementary jump up or down over some c_k

2) $\ell \geq k+1$ if x_ℓ jumps up above c_k, and $\ell \leq k$ if it jumps down below c_k.

We omit the straightforward proof of the claim.

Step 3 We pick an arbitrary (strict) profile $\widetilde{x}$ and a corresponding sequence σ from some x^* to $\widetilde{x}$ as in Step 2, then we check that in each step of the sequence the sum $\sum_1^n g_c(x_\ell) - \mathcal{W}(x)$ cannot decrease, which will conclude the proof that $g_c \in G^+$. This sum is

$$\overbrace{(\sum_{\ell=1}^{n} \mathcal{W}(x_\ell, c))}^{B} - \overbrace{\mathcal{W}(x)}^{C} - \overbrace{\sum_{k=1}^{n-1} \mathcal{W}(c_k, c)}^{D}$$

In the jump up $x_\ell < c_k < x'_\ell$ the net changes are

$$\Delta B = w_k(x'_\ell) - w_{k+1}(x_\ell) + w_{k+1}(c_k) - w_k(c_k) \ ; \ \Delta C = w_\ell(x'_\ell) - w_\ell(x_\ell) \ ; \ \Delta D = 0$$

With the notation $\Delta f(a \to b) = f(b) - f(a)$ and some rearranging this gives

$$\Delta B - \Delta C + \Delta D = \Delta(w_k - w_\ell)(c_k \to x'_\ell) + \Delta(w_{k+1} - w_\ell)(x_\ell \to c_k)$$

where both final terms are non negative because $\ell \geq k+1$ and the opposite of inequality (21) means that $w_k - w_\ell$ and $w_{k+1} - w_\ell$ increase weakly.

The proof for a jump down step is quite similar by computing the variation of $\sum_1^n g_c(x_\ell) - \mathcal{W}(x)$ to be $\Delta(w_\ell - w_k)(c_k \to x_\ell) + \Delta(w_\ell - w_{k+1})(x'_\ell \to c_k)$.

We have shown that g_c is an upper guarantee of $\mathcal{W}$ (even when the sequence c is not strict).

Step 4 Pick now an arbitrary weakly decreasing sequence $c_1 \geq c_2 \geq \cdots \geq c_{n-1}$ and $x_i \in [L, H]$ located in the sequence as $c_k \geq x_i \geq c_{k+1}$ for some $k \in [n] \cup \{0\}$ with the convention $c_0 = H, c_n = L$. It is then straightforward to check that $(x_i; c)$ is a contact point of g_c.

Statement *ii)* We fix g a minimal upper guarantee of $\mathcal{W}$ and recall the notation $\mathcal{C}(g)$ for the set of contact profiles of g (Lemma 4). For each $k \in [n]$ its projection $\mathcal{C}_k(g)$ is the set of those $x_i \in [L, H]$ appearing in some profile $x \in \mathcal{C}(g)$ with the rank k; it is closed because $\mathcal{C}(g)$ is closed and we call its lower bound c_k. The sequence $\{c_k\}$ decreases weakly because in a contact profile where c_k is k-th the type x_{k+1} ranked $k+1$ is weakly below c_k. And $c_n = L$ because c_n is in some contact profile of g.

Check first that $\mathcal{C}_1(g) = [c_1, H]$ with the help of Lemma 7. For each $x_1 \in [c_1, H[$ where g is differentiable and x_1 appears with rank k in some contact profile we have $\frac{dg}{dx}(x_1) = \frac{dw_k}{dx}(x_1) \leq \frac{dw_1}{dx}(x_1)$ because $\mathcal{W}$ is submodular (Lemma 9). This implies $g(x_1) \leq g(c_1) + w_1(x_1) - w_1(c_1)$ everywhere in $[c_1, H]$.

Pick a profile $(c_1, x_{-1}) \in \mathcal{C}(g)$ where c_1 is ranked first and combine the latter inequality with this contact equation:

$$g(c_1) - w_1(c_1) = \sum_2^n (w_k(x_k) - g(x_k)) \geq g(x_1) - w_1(x_1)$$

The inequality above must be an equality because g is an upper guarantee therefore $\frac{dg}{dx}(x_1) = \frac{dw_1}{dx}(x_1)$ a.e. in $[c_1, H]$ and $[c_1, H] = \mathcal{C}_1(g)$.

We repeat this argument for $x_2 \in [c_2, c_1[$. In any of its contact profiles its rank is at least 2 by definition of c_1, so when g is differentiable at x_2 we have $\frac{dg}{dx}(x_2) = \frac{dw_k}{dx}(x_2) \leq \frac{dw_2}{dx}(x_2)$ by submodularity of $\mathcal{W}$. Then $g(x_2) \leq g(c_2) + w_2(x_2) - w_2(c_2)$ holds in $[c_2, c_1]$ and by plugging as above this inequality at a contact profile where c_2 is ranked second, we see that it is an equality and conclude that first, $\frac{dg}{dx}(x_2) = \frac{dw_2}{dx}(x_2)$ a.e. in $[c_2, c_1]$ and second, $[c_2, c_1] \subseteq \mathcal{C}_2(g)$.[8]

The clear induction argument gives $\frac{dg}{dx}(x_k) = \frac{dw_k}{dx}(x_k)$ a.e. in $[c_k, c_{k-1}]$; together with the continuity of g it implies that g is entirely determined by the value $g(L)$. But for $c = (c_1, \cdots, c_{n-1})$ the minimal upper guarantee g_c defined by (23) (and discussed in the proof of statement i)) meets precisely the same differential system therefore g and g_c differ by a constant; if they don't coincide g is either not an upper guarantee or not minimal. ■

9.2 Example 2.C

We can without loss assume that F is the identity because the change of variable $y_i = F(x_i)$ reaches precisely that problem (exactly like in Example 1.A).

The proof resembles that of Proposition 1. Fix a minimal upper guarantee $g^+ \in \mathcal{G}_k^+$ and recall that g^+ is weakly increasing (Lemma 3). Define $p = ng^+(L)$: from $una(x_i) = \frac{1}{n}x_i$ and inequality (6) after Lemma 1 we get $p \geq L$. Observe next that $g_H(x_i) \equiv \frac{1}{n}H$ is in G_k^+ (in fact also in $\mathcal{G}_k^+$ as we show below); if $p > H$ then g^+ is everywhere larger than g_H, a contradiction. So $p \in [L, H]$.

Apply now the feasibility inequality (4) to g^+ at the profile $(\overset{n-k}{L}, \overset{k}{x_i})$:

$$\frac{n-k}{n} p + kg^+(x_i) \geq x_i \text{ for } x_i \in [L, H]$$

[8]Note that $\mathcal{C}_2(g)$ can extend beyond c_1 but this can only happen if $\frac{dw_2}{dx} = \frac{dw_1}{dx}$ in the overlap interval. To see this compare two contact profiles x and y such that $x^1 \geq x^2 > y^1 \geq y^2$ and use the upper bound property at the two profiles where x^2 and y^2 have been swapped plus submodularity of $\mathcal{W}$ to deduce that they are contact profiles as well.

If $k = n$ this gives $g^+(x_i) \geq una(x_i)$: as $una \in g^+$ we conclude $g^+ = una$. For $k \leq n-1$ we combine the inequality above with $g^+(x_i) \geq \frac{1}{n}p$ and obtain

$$g^+(x_i) \geq \max\{\frac{1}{n}p, \frac{1}{k}(x_i - \frac{n-k}{n}p)\} = \frac{1}{n}p + \frac{1}{k}(x_i - p)_+$$

It remains to check that the function on the right, which we write g_p^+, is itself an upper guarantee. Pick an arbitrary profile $x \in [L, H]^{[n]}$ and suppose that p is s. t. $x^\ell \geq p \geq x^{\ell+1}$. We must show

$$\sum_{i=1}^{n} g_p^+(x_i) = p + \frac{1}{k}((\sum_{t=1}^{\ell} x^t) - \ell p) \geq x^k$$

If $p \geq x^k$ we are done because the term in parenthesis is non negative. Assume now $p < x^k$ so that $x^k \geq \cdots \geq x^\ell \geq p \geq x^{\ell+1}$, then note that $(\sum_{t=1}^{\ell} x^t) - \ell p \geq k(x^k - p)$ and we are done.

The proof that for $k \geq 2$ the set $\mathcal{G}_k^-$ is also parametrised by $q \in [L, H]$ as

$$g_p^-(x_i) \geq \frac{1}{n}q + \frac{1}{n-k+1}(x_i - q)_-$$

and for $k = 1$ contains only una, is entirely similar. ■

9.3 Lemma 11

Statement i) is clear because $\mathcal{W}$ is symmetric. In *Statement ii)* upper-hemi-continuity of φ is clear because $\mathcal{W}$ and g are both continous (Lemma 5).

To check that φ is convex valued we fix $(x_1, x_2), (x_1, x_2') \in \Gamma(\varphi)$ and z s. t. $x_2 < z < x_2'$, and check that $\Gamma(\varphi)$ contains (x_1, z) too. Pick some $w \in \varphi(z)$: if $w > x_1$ we see that $\Gamma(\varphi)$ contains (x_1, x_2) and (w, z) s.t. $(x_1, x_2) \ll (w, z)$ which is a contradiction by Lemma 10. If $w < x_1$ we use instead (w, z) and (x_1, x_2') to reach a similar contradiction, and we conclude $w = x_1$.

The proof below that φ is single-valued a. e. will complete that of statement *ii)*.

Statement iii) If $x_1 < x_1'$ in $\mathcal{X}$ and $\varphi^-(x_1) < \varphi^+(x_1')$ we again contradict the strict supermodularity of $\mathcal{W}$ (Lemma 10) . So $x_1 < x_1' \Longrightarrow \varphi^-(x_1) \geq \varphi^+(x_1')$ and φ^- and φ^+ are weakly decreasing.

If $\varphi(x_1)$ is not a singleton, $\varphi^+(x_1) > \varphi^-(x_1)$, then φ^+ jumps down at x_1; a weakly decreasing function can only do this a countable number of times. That the u.h.c. closure of φ^+ contains $[\varphi^-(x_1), \varphi^+(x_1)]$ follows from $\varphi^-(x_1) \geq \varphi^+(x_1 + \delta)$ for any $\delta > 0$.

Statement iv) If $\varphi(L)$ does not contain H we pick some x_1 in $\varphi(H)$: by statement *i*) $\varphi(x_1)$ contains H therefore $x_1 > L$; we reach a contradiction again from Lemma 10 because $\Gamma(\varphi)$ contains $(L, \varphi^+(L))$ and the strictly larger (x_1, H).

Statement v) Kakutani's theorem implies that at least one fixed point exists. If $\Gamma(\varphi)$ contains both (a, a) and (b, b) we contradicts again Lemma 10. Check finally that the inequalities $\varphi^-(a) < a < \varphi^+(a)$ are not compatible. Pick $\delta > 0$ s.t. $\varphi(a)$ contains $a - \delta$ and $a + \delta$: then $\Gamma(\varphi)$ contains $(a, a + \delta)$ and $(a - \delta, a)$ (by symmetry) and we invoke Lemma 10 again. ■

9.4 Theorem 2

Step 0: the integral in (26) is well defined.

For any correspondence φ as in Lemma 11 the integral $\int_a^{x_1} \partial_1 \mathcal{W}(t, \varphi(t))dt$ is the value of $\int_a^{x_1} \partial_1 \mathcal{W}(t, f(t))dt$ for any single-valued selection f of φ: this is independent of the choice of f because φ is multi-valued only at a countable number of points and every single-valued selection of $\varphi(x_1)$ is a measurable function.

Statement ii) Fix $g \in \mathcal{G}^-$ and its contact correspondence φ. The function $\mathcal{W}$ is uniformly Lipschitz in $[L, H]^2$ so by Lemma 7 g is Lipschitz as well, hence differentiable a. e.. The derivative $\frac{dg}{dx}$ is given by property (10) in the Corollary to Lemma 7:

$$\frac{dg}{dx}(x_1) = \partial_1 \mathcal{W}(x_1, x_2) \text{ for any } x_2 \in \varphi(x_1)$$

therefore we can write the RH as $\partial_1 \mathcal{W}(x_1, \varphi(x_1))$ without specifying a particular selection of $\varphi(x_1)$.

Note that $g(a) = una(a)$ because $(a, a) \in \Gamma(\varphi)$. Now integrating the differential equation above with this initial condition at a gives the desired representation (26).

Statement *i*)

Step 1 Lemma 11 implies that $\Gamma(\varphi)$ is a one-dimensional line connecting (L, H) and (H, L) that we can parametrise by a smooth mapping $s \to (\xi_1(s), \xi_2(s))$ from $[0, 1]$ into $[L, H]^2$ s.t. $\xi_1(\cdot)$ increases weakly from L to H and $\xi_2(\cdot)$ decreases weakly from H to L. We can also choose this mapping so that $\xi_1(\frac{1}{2}) = \xi_2(\frac{1}{2}) = a$, the fixed point of φ.[9]

[9]If a is 0,or 1 we check that (26) defines the two canonical stand alone guarantees in Proposition 3.

We fix an arbitrary selection γ of φ, an arbitrary $\overline{x}_1$ in $[L, H]$, and check the identity

$$\int_a^{\overline{x}_1} \partial_1 \mathcal{W}(t, \varphi(t))dt + \int_a^{\gamma(\overline{x}_1)} \partial_1 \mathcal{W}(t, \varphi(t))dt = \mathcal{W}(\overline{x}_1, \gamma(\overline{x}_1)) - \mathcal{W}(a, a) \quad (34)$$

We change the variable t to s by $t = \xi_1(s)$ in the former and by $t = \xi_2(s)$ in the latter. Next $\overline{s}$ is the parameter at which $(\xi_1(\overline{s}), \xi_2(\overline{s})) = (\overline{x}_1, \gamma(\overline{x}_1))$ and we rewrite the LH above as

$$\int_{\frac{1}{2}}^{\overline{s}} \partial_1 \mathcal{W}(\xi_1(s), \xi_2(s)) \frac{\partial \xi_1}{\partial s}(s)ds + \int_{\frac{1}{2}}^{\overline{s}} \partial_1 \mathcal{W}(\xi_2(s), \xi_1(s)) \frac{\partial \xi_2}{\partial s}(s)ds$$

where in each term $\partial_1 \mathcal{W}(t, \varphi(t))$ we can select a proper selection of the (possible) interval because $(\xi_1(s), \xi_2(s)) \in \Gamma(\varphi)$. As $\mathcal{W}(x_1, x_2)$ is symmetric in x_1, x_2, we can replace the second integral by $\int_{\frac{1}{2}}^{\overline{s}} \partial_2 \mathcal{W}(\xi_1(s), \xi_2(s)) \frac{\partial \xi_2}{\partial s}(s)ds$ and conclude that the sum is precisely

$$\mathcal{W}(\xi_1(\overline{s}), \xi_2(\overline{s})) - \mathcal{W}(\xi_1(\frac{1}{2}), \xi_2(\frac{1}{2})) = \mathcal{W}(\overline{x}_1, \gamma(\overline{x}_1)) - \mathcal{W}(a, a)$$

Step 2 We show that (26) defines a bona fide guarantee g: $g(x_1) + g(x_2) \leq \mathcal{W}(x_1, x_2)$ for all $x_1, x_2 \in [L, H]$.

The identity (34) amounts to $g(x_1) + g(\gamma(x_1)) = \mathcal{W}(x_1, \gamma(x_1))$ for all x_1. If we prove that $g \in \boldsymbol{G}^-$ this will imply it is maximal. Compute

$$g(x_1)+g(x_2) = \mathcal{W}(x_1, \gamma(x_1))+g(x_2)-g(\gamma(x_1)) = \mathcal{W}(x_1, \gamma(x_1))+\int_{\gamma(x_1)}^{x_2} \partial_1 \mathcal{W}(t, \varphi(t))dt$$

We are left to show

$$\int_{\gamma(x_1)}^{x_2} \partial_1 \mathcal{W}(t, \varphi(t))dt \leq \mathcal{W}(x_1, x_2) - \mathcal{W}(x_1, \gamma(x_1)) \quad (35)$$

We assume without loss $x_1 \leq x_2$ and distinguish several cases by the relative positions of a and x_1, x_2 .

Case 1: $a \leq x_1 \leq x_2$, so that $\gamma(x_1) \leq a$. For every $t \geq \gamma(x_1)$ property *iii*) in Lemma 11 implies $\varphi^+(t) \leq \varphi^-(\gamma(x_1))$ and $\varphi(\gamma(x_1))$ contains x_1: therefore submodularity of $\mathcal{W}$ implies $\partial_1 \mathcal{W}(t, \varphi(t)) \leq \partial_1 \mathcal{W}(t, x_1)$ and

$$\int_{\gamma(x_1)}^{x_2} \partial_1 \mathcal{W}(t, \varphi(t))dt \leq \int_{\gamma(x_1)}^{x_2} \partial_1 \mathcal{W}(t, x_1)dt = \mathcal{W}(x_2, x_1) - \mathcal{W}(\gamma(x_1), x_1)$$

Case 2: $x_1 \leq a \leq \gamma(x_1) \leq x_2$. Similarly for $t \geq \gamma(x_1)$ we have $\varphi^+(t) \leq \varphi^-(\gamma(x_1))$ and conclude as in Case 1.

Case 3: $x_1 \leq x_2 \leq a$, so that $\gamma(x_1) \geq a$. For all $t \leq \gamma(x_1)$ we have $\varphi^-(t) \geq \varphi^+(\gamma(x_1))$ and $\varphi(\gamma(x_1))$ contains x_1: now submodularity of $\mathcal{W}$ gives $\partial_1 \mathcal{W}(t, z) \geq \partial_1 \mathcal{W}(t, x_2)$ for z between x_2 and $\gamma(x_1)$ and the desired inequality because the integral in (35) goes from high to low.

Case 4: $x_1 \leq a \leq x_2 \leq \gamma(x_1)$. Same argument as in Case 3. ■

9.5 Statement *iii*) in Proposition 5

We set $A = \{a, b\}$ so$\mathcal{X} = [L_a, H_a] \times [L_b, H_b]$ and fix $g \in \mathcal{G}^+$ throughout.
Step 1. The function $\mathcal{W}_A$ isuniformly 1-Lipschitz for the ℓ_∞ norm and so is g (Lemma 7). As discussed just after the proof of the latter, we know that along any coordinate x_{ia} both g and $\mathcal{W}_A$ are a.e. differentiable; both are also weakly increasing (Lemma 3). Fix $\overline{x}_1 \in [L, H]$ s. t. $\overline{x}_{1a} \in]L_a, H_a[$ and a corresponding contact profile $(\overline{x}_1, \widetilde{x}_{-1})$ of g. If $x_{1a} \to \mathcal{W}_A((x_{1a}, \overline{x}_{1b}), \widetilde{x}_{-1})$ has a kink at $\overline{x}_{1a}$ its left derivative is 0 and its right one is 1: then the inequalities (9) imply that g is not differentiable in x_{1a} at $\overline{x}_1$. So when g *is* differentiable in x_{1a} at $\overline{x}_1$, which is true a. e. in x_{1a}, so is $\mathcal{W}_A$ and $\frac{\partial g}{\partial x_{1a}}(\overline{x}_1) = \frac{\partial \mathcal{W}_A}{\partial x_{1a}}(\overline{x}_1, \widetilde{x}_{-1})$ is 0 or 1.

If $\frac{\partial g}{\partial x_{1a}}(x_{1a}, \overline{x}_{1b}) = 0$ we claim that the same is true for any smaller $x'_{1a} \in]L_a, x_{1a}[$. When we lower x_{1a} in the contact equality (7)

$$g(x_{1a}, \overline{x}_{1b}) + \sum_{j=2}^{n} g(\widetilde{x}_j) = \mathcal{W}_A((x_{1a}, \overline{x}_{1b}), \widetilde{x}_{-1})$$

the RH does not change while the LH does not increase (Lemma 3); but if the LH decreases we reach a contradiction of the upper guarantee inequality (4): therefore $g(\cdot, \overline{x}_{1b})$ is flat as claimed.

If instead $\frac{\partial g}{\partial x_{1a}}(x_{1a}, \overline{x}_{1b}) = 1$ and we increase x_{1a} to $x'_{1a} \in]x_{1a}, H_a[$ in the contact equality, the RH term increases at speed 1 while the LH term increases with speed *at most* 1; but the LH term cannot fall below the RH one because g is an upper guarantee, so we see that g increases at speed 1 in $]x_{1a}, H_a[$. Because the two subsets of $]L_a, H_a[$ just described cover all but a subset of measure zero and g is continuous (Lemma 4), we conclude that there is a critical value $\tau \in [L_a, H_a]$ and a number $\pi = g(L_a, \overline{x}_{1b})$, both depending on $\overline{x}_{1b}$ such that

$$g(x_{1a}, \overline{x}_{1b}) = (x_{1a} - \tau)_+ + \pi \text{ for } x_{1a} \in [L_a, H_a] \quad (36)$$

Step 2. We note that a contact profile of g has one agent (H_a, H_b) and $n-1$ others with type (L_a, L_b). This is because in any contact equation (7) containing (H_a, H_b):

$$g(H_a, H_b) + \sum_{j=2}^{n} g(x_j) = H_a + H_b$$

If we lower each x_j to (L_a, L_b) the RH does not change while the LH sum must stay flat (same argument as in Step 1). Setting $\mu = g(L_a, L_b)$ we conclude

$$g(H_a, H_b) = H_a + H_b - (n-1)\mu \tag{37}$$

Step 3 We apply equation (36) successively to $\overline{x}_{1b} = L_b$ and $\overline{x}_{1a} = L_a$: this produces two numbers α, β s. t. for all $x_1 \in [L, H]$

$$g(x_{1a}, L_b) = (x_{1a} - \alpha)_+ + \mu \text{ and } g(L_a, x_{1b}) = (x_{1b} - \beta)_+ + \mu \tag{38}$$

where $(\alpha, \beta) \in [L, H]$.

Suppose that $g(\alpha, z_b) = \mu$ for some $z_b > \beta$: it implies $g(L_a, z_b) = \mu$ but the RH equation in (38) says $g(L_a, z_b) = \mu + (z_b - \beta)$, contradicting the definition of β. So equation (36) for $\overline{x}_{1a} = \alpha$ uses some $\beta^* \leq \beta$:

$$g(\alpha, x_{1b}) = (x_{1b} - \beta^*)_+ + \mu \text{ for } x_{1b} \in [L_b, H_b]$$

(recall $g(\alpha, L_b) = \mu$). Similarly there is some $\alpha^* \leq \alpha$ such that

$$g(x_{1a}, \beta) = (x_{1a} - \alpha^*)_+ + \mu \text{ for } x_{1a} \in [L_a, H_a]$$

The two last equations imply

$$g(\alpha, \beta) - \mu = \beta - \beta^* = \alpha - \alpha^* = \delta \geq 0 \tag{39}$$

Step 4 We assume $\delta > 0$ and derive a contradiction.

In the rectangle[10] $[(\alpha, \beta^*), (H_a, \beta)]$ we know from step 3 that $\frac{\partial g}{\partial x_{1b}} \equiv 1$ from (α, β^*) to (α, β) and $\frac{\partial g}{\partial x_{1a}} \equiv 1$ from (α, β) to (H_a, β). On the alternative path (α, β^*) to (H_a, β^*) then to (H_a, β), the corresponding derivatives are at most 1 therefore they are 1 everywhere. So the derivative of g is 1 on the edges of $[(\alpha, \beta), (H_a, \beta)]$ and $[(\alpha, \beta), (\alpha, H_b)]$.

Apply now step 1 to $\overline{x}_{1a} = H_a$: $g(H_a, \cdot)$ takes the form (36) on $[L_b, H_b]$. We just showed that $\frac{\partial g}{\partial x_{1b}}(H_a, \cdot) \equiv 1$ in the non trivial interval $[\beta^*, \beta]$ therefore it is still 1 in $[(H_a, \beta), H]$. Now we see that the derivative of g is 1 on

[10] If $\alpha = L_a$ or H_a this is an interval but this does not affect the argument to show that δ must be zero.

the four edges of the rectangle $[(\alpha,\beta),H]$. A similar argument shows that it is 1 as well on any horizontal or vertical interval splitting $[(\alpha,\beta),H]$ in two.

From $\frac{\partial g}{\partial x_{1a}} \equiv \frac{\partial g}{\partial x_{1b}} \equiv 1$ in $[(\alpha,\beta),H]$ we get

$$g(H) - g(\alpha,\beta) = H_a + H_b - (\alpha+\beta)$$

which, combined with equations (37) and (39), gives $\alpha+\beta = n\mu+\delta$. However if we apply the upper guarantee inequality (4) to the profiles $(\alpha,\beta^*),(\alpha^*,\beta)$ and $n-2$ copies of (L_a,L_b) we have

$$n\mu = g(\alpha,\beta^*) + g(\alpha^*,\beta) + (n-2)\mu \geq \alpha+\beta$$

and we conclude $\alpha+\beta = n\mu$ and $\delta = 0$, the desired contradiction.

We have shown: $\alpha = \alpha^*$, $\beta = \beta^*$, $\alpha+\beta = n\mu$, and $g(\alpha,\beta) = \mu$ (by (39)).

Step 5 We now compute $\frac{\partial g}{\partial x_{1a}}, \frac{\partial g}{\partial x_{1b}}$ everywhere in $[L,H]$ to conclude the proof. We just showed that both derivatives equal 1 in $[(\alpha,\beta),H]$; they are both 0 in $[L,(\alpha,\beta)]$ because $g(\alpha,\beta) = \mu$. In the remaining rectangle $[(\alpha,L_b),[(H_a,\beta)]$ we have $\frac{\partial g}{\partial x_{1a}} \equiv 1$ on the two edges parallel to the x_a-axis, and $\frac{\partial g}{\partial x_{1b}} \equiv 0$ on the edge $[(\alpha,L_b),(\alpha,\beta)]$. So g is constant between (H_a,L_b) and (H_a,β), and also between any (x_a,L_b) and (x_a,β) by the usual integration argument. So $\frac{\partial g}{\partial x_{1a}} \equiv 1, \frac{\partial g}{\partial x_{1b}} = 0$ holds in the entire rectangle. The symmetric property holds in the rectangle $[(L_a,\beta),[(\alpha,H_b)]$.

The function g is now entirely determined by the choice of (α,β) in $[L,H]$ because $g(L) = \mu = \frac{1}{n}(\alpha+\beta)$. Its closed form expression is

$$\text{for all } x_1 \in [L,H] : g(x_1) = (x_{1a}-\alpha)_+ + (x_{1b}-\beta)_+ + \mu =$$

$$= (x_{1a}-\alpha)_+ + \frac{1}{n}\alpha + (x_{1b}-\beta)_+ + \frac{1}{n}\beta = g^+_\alpha(x_{1a}) + g^+_\beta(x_{1b})_+$$

■